Degenerate parabolic operators of Kolmogorov type with a geometric control condition

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Abstract

We consider Kolmogorov-type equations on a rectangle domain $(x, v) \in \Omega = \mathbb{T} \times (-1, 1)$, that combine diffusion in variable v and transport in variable x at speed v^γ, $\gamma \in \mathbb{N}^*$, with Dirichlet boundary conditions in v. We study the null controllability of this equation with a distributed control as source term, localized on a subset ω of Ω,

$$\begin{cases} \left(\partial_t - v^\gamma \partial_x - \partial_v^2\right) f(t,x,v) = u(t,x,v) 1_\omega(x,v)\,, & (t,x,v) \in (0,T) \times \mathbb{T} \times (-1,1)\,, \\ f(t,x,\pm 1) = 0\,, & (t,x) \in (0,T) \times \mathbb{T}\,. \end{cases}$$

Thanks to the interplay between diffusion in v and transport in x, the equation diffuses both in variables v and x (contrarily to equation $(\partial_t - \partial_v^2) g(t,x,v) = 0$) but, in a weaker way than the $2D$ heat equation (i.e. $(\partial_t - \partial_x^2 - \partial_v^2) g(t,x,v) = 0$). Thus, natural questions are the following ones.

Question 1: Is the diffusion in variable v strong enough for observability to hold when the control acts on a horizontal strip $\omega = \mathbb{T} \times (a, b)$ with $0 < a < b < 1$, whatever $\gamma \in \mathbb{N}^*$ is? (i.e. as for equation $(\partial_t - \partial_v^2) g = 0$, $(t, x, v) \in (0, T) \times \mathbb{T} \times (-1, 1)$)

Question 2: Is the diffusion in variable x sufficient for null controllability to hold when the control acts on a vertical strip $\omega = \omega_1 \times (-1, 1)$ where $\omega_1 \subset\subset \mathbb{T}$? (i.e. as for the 2D heat equation)

When the control acts on a horizontal strip $\omega = \mathbb{T} \times (a, b)$ with $0 < a < b < 1$, then the system is null controllable in any time $T > 0$ when $\gamma = 1$, and only in large time $T > T_{min} > 0$ when $\gamma = 2$: a finite speed of propagation occurs (see [1]). When $\gamma > 3$, the system is not null controllable (whatever T is) in this configuration, even if unique continuation holds (see [2]). Thus, the first order term $v^\gamma \partial_x$ weakens strongly the diffusion in variable v when $\gamma \geqslant 3$. These results answer **Question 1**.

When the control acts on a vertical strip $\omega = \omega_1 \times (-1, 1)$ with $\overline{\omega_1} \subset \mathbb{T}$, we investigate the null controllability on a toy model, where $(\partial_x, x \in \mathbb{T})$ is replaced by $((-\Delta)^{1/2}, x \in \Omega_1)$, and Ω_1 is an open subset of $\mathbb{R}^N$. As the original system, this toy model satisfies the controllability properties listed above. We prove that, for $\gamma = 1, 2$ and for appropriate domains (Ω_1, ω_1), then null controllability does not hold (whatever $T > 0$ is), when the control acts on a vertical strip $\omega = \omega_1 \times (-1, 1)$ with $\overline{\omega_1} \subset \Omega_1$ (see [2]). Thus, a geometric control condition is required for the null controllability of this toy model. It indicates that a geometric control condition may be necessary for the original model too. This is a conjecture about the answer of **Question 2**.

References

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