Introduction to stochastic equations and some links with control theory

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Introduction and motivations

Modeling in physics, biology or chemistry often involves PDEs with random coefficients

$$\frac{\partial u}{\partial t} = F(u, \nabla u, D^2 u, ...) + g(u)\xi + \eta$$

with $\xi(t, x, \omega)$ and $\eta(t, x, \omega)$ random parameters

Natural assumptions : (characteristic scales $\ll 1$)

- independence at different times (sometimes for different x)
- stationarity of the law
- zero mean
- not "too much" irregularity

$\rightsquigarrow$ ξ, η are Gaussian with

$$\mathbf{E}(\xi(t,x)\xi(s,y)) = \delta_{t-s}\delta_{x-y} \text{ or } \delta_{t-s}c(x-y)$$

Examples :

- Reaction-diffusion equation (dynamic phase transitions, pattern formation)

$$\frac{\partial u}{\partial t} = \Delta u + f(u) + \varepsilon g(u)\xi$$

with e.g. $g(u) = 1$ or $g(u) = \sqrt{u(1-u)}$
metastability, random traveling waves, large deviations, ...

- Stochastic Navier-Stokes equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u.\nabla)u + \nabla p = \varepsilon\xi \\ \mathrm{div} u = 0 \end{cases}$$

Invariant measure, ergodicity...

- Stochastic conservation laws (fluid vorticity)

$$\frac{\partial u}{\partial t} + div(A(u) + g(u)\xi + \eta) = 0$$

stabilization by noise, invariant measure...

- Stochastic nonlinear Schrödinger equations (nonlinear optics, molecular systems, Bose Einstein condensation)

$$i\frac{\partial u}{\partial t} + \Delta u + V(x)u + |u|^2 u = \varepsilon g(u,x)\xi$$

with $g(u) = 1$ or $g(u) = u$, or $g(u) = x^2 u$

Dispersion managed optical fibers :

$$i\partial_z u + \varepsilon\eta\Delta u + |u|^2 u = 0$$

with $\eta(z,\omega)$ such that $\mathbf{E}(\eta(z)\eta(z')) = \delta_{z-z'}$.

Stabilization by noise ; influence on blow-up phenomena ; computation of transmission errors.

Note that :

- $\xi(t,x,\omega)$ is not well defined, but $\int_0^t \xi(s,x,\omega)ds$ is (or $\int_0^t \int_o^x \xi(s,y,\omega)dsdy$ if no space correlation)
- Need to define the product $g(u)\xi \rightsquigarrow \int_0^t g(u(s))\xi(s)ds$
- Need to solve the stochastic PDE (will give details only for additive noise i.e. $g(u) = cste$)

Link with control theory : In many situations, replacing the noise by a control term gives informations on the solution. Similar objects appear in Malliavin calculus.

Stochastic calculus in dimension 1

Brownian motion

Brownian motion (standard, nomalized)

$B = (B_t)_{t \geq 0}$ family of real random variables on $(\Omega, \mathcal{F}, \mathbf{P})$ such that

- $B(0) = 0$
- for a.e. $\omega \in \Omega$, $B(., \omega)$ is continuous in time
- any family of increments $(B(t_1) - B(t_0), ..., B(t_n) - B(t_{n-1})) = (X_1, ..., X_n)$ is an independent family (which is equivalent to $\mu_{(X_1,..,X_n)} = \mu_{X_1} \otimes ... \otimes \mu_{X_n}$)

This implies that $B(t)_{t \geq 0}$ is a Gaussian process and $B(t) - B(s)$ is centered with variance $|t - s|$

Remarks :

- If $\mathcal{F}_t = \sigma\{B(s), 0 \le s \le t\}$, then $(\mathcal{F}_t)_t$ is a nondecreasing family of σ-algebras. A random variable is $\mathcal{F}_t$-measurable if it "depends only on the past"
- B may be obtained as the rescaled limit of random walks (functional central limit theorem)
- Let $(Y_k)_{k\ge 0}$ be a family of independent real valued $\mathcal{N}(0,1)$; then
$$B(t) = \frac{t}{\sqrt{\pi}} Y_0 + \sqrt{\frac{2}{\pi}} \sum_{k\ge 1} \frac{\sin kt}{k} Y_k$$
is a normalized B.M. on $[0,\pi]$ (the series converges in $L^2(\Omega)$). Formally :
$$\dot{B}(t) = \frac{1}{\sqrt{\pi}} Y_0 + \sqrt{\frac{2}{\pi}} \sum_{k\ge 1} \sin kt Y_k$$
- Formally recover the δ correlation of $\dot{B}$: $\mathbf{E}(\dot{B}(t)\dot{B}(s)) = \delta_{t-s}$

Regularity :

- Since $B(t) - B(s) \sim \mathcal{N}(0, |t-s|)$ we deduce

$$\mathbf{E}(|B(t) - B(s)|^{2k}) \leq C_k \mathbf{E}(|B(t) - B(s)|^2)^k \leq C_k |t-s|^k$$

this implies (Kolmogorov criterion) that $B \in C^\alpha([0,T])$ for any $\alpha < 1/2 - 1/k$ hence for any $\alpha < 1/2$.

Regularity :

- Note that

$$\mathbf{E}\sum_{i=0}^{n-1}(B(t_{i+1})-B(t_i))^2 = \sum_{i=0}^{n-1}\mathbf{E}(B(t_{i+1})-B(t_i))^2 = \sum_{i=0}^{n-1} t_{i+1}-t_i = T$$

- One can prove (independence of increments) that

$$\lim \sum_{i=0}^{n-1}(B(t_{i+1})-B(t_i))^2 = T$$

in $L^2(\Omega)$ (or *a.e.*) as the subdivision step goes to zero.

- In particular, for $\alpha \geq 1/2$, the probability that $B \in C^\alpha([0,T])$, or $B \in BV(0,T)$, is equal to zero.

One dimensional Itô integral

We want to give a meaning to : $\frac{dx}{dt} = f(x) + g(x)\dot{B}(t)$

$\rightsquigarrow$ we need to define " $\displaystyle\int_0^t g(x)\frac{dB}{dt}dt$" $= \displaystyle\int_0^t g(x)dB$ which is not a Stieljes integral.

Let $g \in C([0,T]; L^2(\Omega))$ such that $g(t)$ is $\mathcal{F}_t$-measurable for $t \in [0,T]$. We set

$$\int_0^T g(s)dB(s) = \lim_{|\sigma|\to 0} \sum_{i=0}^{n-1} g(t_i)(B(t_{i+1}) - B(t_i))$$

We note that :

- the expectation of the right hand side is equal to zero
- the variance of the right hand side is equal to

$$\sum_{i=0}^{n-1} \mathbf{E}(g^2(t_i))(t_{i+1} - t_i) = \int_0^T \mathbf{E}(\bar{g}^2(s))ds$$

where $\bar{g}$ is the piecewise constant process on the subdivision.

It follows : The Itô integral is defined by extending the isometry to "adapted" processes in $C([0,T];L^2(\Omega))$ (and actually for more general processes) with

- $\mathbf{E}(\int_0^T g(s)dB(s)) = 0$
- $\mathbf{E}\left((\int_0^T g(s)dB(s))^2\right) = \int_0^T \mathbf{E}(g^2(s))ds$

Notation : the equality $dX = fdt + gdB$ means that for any $t \in [0,T]$,

$$X(t) = X(0) + \int_0^t f(s)ds + \int_0^t g(s)dB(s)$$

1-D Itô formula : if $F \in C_b^2(\mathbf{R})$, then

$$dF(X) = F'(X)dX + \frac{1}{2}F''(X)g^2dt$$

Multi-dimensional case : B is a Brownian motion with values in $\mathbf{R}^n$ iff $B = (B_1, ..., B_n)$ with (B_i) family of independent B.M.

Stochastic calculus in infinite dimension

Infinite dimensional Wiener process :

Let

- H be a separable Hilbert space (typically, $H = L^2(O)$ or $H = L^2(\mathbf{R}^d)$)
- $(e_k)_{k\in\mathbf{N}}$ a complete orthonormal system in H
- $(B_k)_{k\in\mathbf{N}}$ a family of independent B.M.

then $W(t) = \sum_{k=0}^{\infty} e_k B_k$ is a cylindrical Wiener process on H.

Remark :

- Formally, $\frac{\partial W}{\partial t} = \sum_{k=0}^{\infty} e_k \dot{B}_k$ is a space-time white noise.
- the series is not convergent in H but if $H \subset U$ with $\sum_k |e_k|_U^2 < \infty$ then the series converges in $L^2(\Omega; U)$
- If $H = L^2(O)$ for some bounded domain O in $\mathbf{R}^d$ then one may choose $U = H^{-s}(O)$ with $s > d/2$.
- the definition is independent of the choice of $(e_k)_{k\in\mathbf{N}}$

Spatially correlated noise :

Let $\Phi : H \to H$ be a bounded linear operator (smoothing) and let

$$\tilde{W}(t) = \sum_{j=0}^{\infty} \Phi e_j B_j = \Phi W$$

The series converges in $L^2(\Omega; H)$ iff Φ is Hilbert-Schmidt in H.

In this case, $\tilde{W}(t)$ is Gaussian with values in H i.e. for any $h \in H$, $(\tilde{W}(t), h)$ is Gaussian

Covariance operator : C such that

$$\mathbf{E}((\tilde{W}(t), h)(\tilde{W}(t), k)) = t(Ch, k)$$

From the independence of the B.M,

$$\mathbf{E}((\tilde{W}(t), h)(\tilde{W}(t), k)) = \sum_{j,l}(\Phi e_j, h)(\Phi e_l, k)\mathbf{E}(B_j(t)B_l(t))$$
$$= t\sum_j(\Phi e_j, h)(\Phi e_l, k) = t\sum_j(e_j, \Phi^* h)(e_l, \Phi^* k) = t(\Phi^* h, \Phi^* k)$$

$\rightsquigarrow C = \Phi\Phi^*$

Remark :
Let $H = L^2(\mathcal{O})$ and $\Phi f(x) = \int_O \mathcal{K}(x,y)f(y)dy = (\mathcal{K}(x,.), f)$ then formally again,

$$\mathbf{E}(\partial_t W(t,x)\partial_s W(s,y)) = \delta_{t-s}c(x,y)$$

with

$$c(x,y) = (\mathcal{K}(x,.), \mathcal{K}(y,.)) = \int_O \mathcal{K}(x,z)\mathcal{K}(y,z)dz$$

for $x, y \in \mathcal{O}$. Thus

- $\Phi = Id \rightsquigarrow c(x,y) = \delta_{x-y}$ space-time white noise ;
- Φ is a Hilbert-Schmidt operator from $L^2(\mathcal{O})$ to $L^2(\mathcal{O})$ iff $\mathcal{K} \in L^2(\mathcal{O} \times \mathcal{O})$
- If Φ is a Hilbert-Schmidt operator from H to some space K then $W(t)$ is $C^{1/2-}(\mathbf{R}^+; K)$

Infinite dimensional stochastic integral :

Let $\Psi \in L^2(0, T; \mathcal{L}(H, K))$ where H and K are separable Hilbert spaces. Let W be a cylindrical Wiener process on H.
Assume that Ψ is adapted (predictible) and

$$\mathbf{E} \int_0^t \|\Psi(s)\|^2_{HS(H,K)} ds < +\infty$$

Then if $(f_j)_{j \in \mathbf{N}}$ is a c.o.s. in K,

$$\int_0^t \Psi(s) dW(s) = \sum_{j,k} \int_0^t (\Psi(s)e_k, f_j)_K dB_k(s) f_j$$

defines a a.s. continuous process with values in K. Moreover

- $\mathbf{E}(\int_0^t \Psi(s) dW(s)) = 0$
- $\mathbf{E}(\| \int_0^t \Psi(s) dW(s) \|^2_K) = \mathbf{E} \int_0^T \|\Psi(s)\|^2_{HS(H,K)} ds$
- $\int_0^t \Psi(s) dW(s)$ is adapted

Itô formula :

Notation :

$$du = f(t)dt + \Psi(t)dW(t)$$

where $f(t)$ is K-valued, $\Psi(t)$ is $HS(H,K)$-valued.

Assume $K = H$ (for simplicity) and let F be a C_b^2 functional from H into $\mathbf{R}$; then for u as above,

$$dF(u) = (F'(u), du) + \frac{1}{2} Tr(F''(u)\Psi\Psi^*)dt$$

which means

$$\begin{aligned} F(u(t)) &= F(u_0) + \textstyle\int_0^t (F'(u(s)), f(s))ds \\ &+ \int_0^t (F'(u(s)), \Psi(s)dW(s)) \\ &+ \tfrac{1}{2} \sum_l \int_0^t (F''(u(s))\Psi(s)e_l, \Psi(s)e_l)ds \end{aligned}$$

Remark on the Stratonovich integral :

One may define a stratonovich integral as (assuming $K = H$)

$$\int_0^t \Psi(s) \circ dW(s) = \lim \sum_l \psi(\frac{t_l + t_{l+1}}{2})(W(t_{l+1}) - W(t_l))$$

- Preserves the chain rules
- Natural in physics (occurs as limit of correlated processes in time)

however

- This integral has bad probabilisitic properties : it average is not zero and it does not define a martingale.

In the case $du = f(u)dt + \Psi(u) \circ dW$ we may rewrite in Itô form :

$$du = (f(u) + G_\Psi(u))dt + \Psi(u)dW$$

with

$$G_\Psi(u) = \frac{1}{2} \sum_l \Psi'(u)(\Psi(u)e_l)e_l \in H$$

The stochastic NLS equations

Let us consider the stochastic NLS equations

$$\begin{cases} i\partial_t u + \Delta u + \lambda |u|^{2\sigma} u = g(u)\xi \\ u(0) = u_0 \end{cases}$$

with $u(t, x, \omega) \in \mathbf{C}$, $x \in \mathbf{R}^d$, and

- either $g(u) = 1$ (additive noise), $\xi(t, x, \omega) \in \mathbf{C}$ and

$$\mathbf{E}(\xi(t,x)\bar{\xi}(s,y)) = c(x,y)\delta_{t-s}$$

  with $c \in L^1(\mathbf{R}^d \times \mathbf{R}^d)$

- or $g(u) = u$ (multiplicative noise, linear in u, Stratonovitch), $\xi(t, x, \omega) \in \mathbf{R}$ and

$$\mathbf{E}(\xi(t,x)\xi(s,y)) = c(x,y)\delta_{t-s}$$

  with $c \in L^1(\mathbf{R}^d \times \mathbf{R}^d)$

Remark : space-time white noise = open problem even for linear equation

Mathematical formulation :

Let $H = L^2(\mathbf{R}^d;\mathbf{C}) \sim L^2(\mathbf{R}^d;\mathbf{R}^2)$, and let

- $(e_m)_{m\in\mathbf{N}}$ Hilbert basis of $L^2(\mathbf{R}^d;\mathbf{C})$,
- $(B_m)_{m\in\mathbf{N}}$ family of independent real valued B.M.

and

$$W(t) = \sum_{m\in\mathbf{N}} \Phi e_m B_m(t)$$

with Φ bounded linear Hilbert-Schmidt operator on H, with the additional property in the multiplicative case : $P_{i\mathbf{R}}\Phi = 0$, where $\mathbf{C} = \mathbf{R} \oplus i\mathbf{R}$; hence Φ is degenerate : $R(\Phi)$ cannot be dense in $L^2(\mathbf{R};\mathbf{C})$

Then equation written as

$$idu + (\Delta u + \lambda |u|^{2\sigma} u)dt = \begin{cases} dW \\ u \circ dW \end{cases}$$

with $\lambda = \pm 1$, $x \in \mathbf{R}^d$, $t \geq 0$.

Remark in the multiplicative case :

Stratonovich product : necessary to preserve the L^2 norm (physical constraint in the model)
Use the rule to change into a Itô equation : let

$$(\Psi(u)v)(x) = -iu(x)(\Phi v)(x), \; u, v \in L^2(\mathbf{R}^d; \mathbf{C}) = H$$

then equation written as

$$du = (i\Delta u + i\lambda |u|^{2\sigma} u)dt + \Psi(u) \circ d\tilde{W}$$

and equivalent Itô form :

$$du = (i\Delta u + i\lambda |u|^{2\sigma} u + G_\Psi(u))dt + \Psi(u)d\tilde{W}$$

with

$$G_\Psi(u) = \frac{1}{2} \sum_m \Psi'(u)(\Psi(u)e_m)e_m = -\frac{u}{2} \sum_m (\Phi e_m)^2 = -\frac{u}{2} F_\Phi$$

since $\Psi'(u)v = \Psi(v)$ i.e. $(\Psi'(u)v)w = -iv\Phi w$

Conservation of the L^2 norm : Formally apply the Itô Formula with $F(u) = |u|_{L^2}^2$, the inner product $(u, v) = \int_{\mathbf{R}^d} u(x)\bar{v}(x)dx$ and the equation

$$du = i(\Delta u + \lambda |u|^{2\sigma} u)dt - iudW - \frac{1}{2} uF_\phi dt$$

then

$$d|u|_{L^2}^2 = (2u, du) + \frac{1}{2} Tr(F''(u)(-iu\Phi)(-iu\Phi)^*)dt$$

i.e.

$$\begin{aligned} |u(t)|_{L^2}^2 &= |u_0|_{L^2}^2 + 2\Re[i \sum_m \int_0^t \int_{\mathbf{R}^d} |u(s)|^2 (\Phi e_m)(x) dx dB_m(s)] \\ &\quad - \Re \int_0^t \int_{\mathbf{R}^d} |u(s)|^2 F_\Phi(x) dx ds + \sum_m \int_0^t (u\Phi e_m, u\Phi e_m) ds \\ &= 0, \ a.s. \end{aligned}$$

Justification : cut-off + regularization, for $u \in C([0, T]; H)$, a.s.

Additive case :

We first consider the linear equation, we write :

$$Z(t) = i\int_0^t e^{i(t-s)\Delta}dW(s)$$

so that

$$idZ + (\Delta Z)dt = dW$$

Then $u = Z + v$ with

$$v(t) = e^{it\Delta}u_0 + i\lambda\int_0^t e^{i(t-s)\Delta}(|v+Z|^{2\sigma}(v+Z))ds$$

Note that Z is Gaussian and by Parseval theorem :

$$\mathbf{E}\|Z(t,x)\|_{L^2}^2 = \mathbf{E}\int_0^t \|e^{i(t-s)\Delta}\Phi\|_{HS(L^2)}^2 ds$$
$$= \sum_m \int_0^t \int_{\mathbf{R}^d} |e^{i(t-s)|\xi|^2}\widehat{\Phi e_m}(\xi)|^2 d\xi = t\sum_m |\Phi e_m|_{L^2}^2 = t\|\Phi\|_{HS(L^2)}^2$$

The same is true is replace $\|Z\|_{L^2}$ by $\|Z\|_{H^s}$ for any Sobolev space $H^s(\mathbf{R}^d)$, with $s \in \mathbf{R}$. Hence

- no hope to solve equation in $H^s(\mathbf{R}^d)$ in the case Φ is a convolution operator (stationary, or homogeneous noise in space)
- In particular, no hope to solve equation with space-time white noise
- even if the noise is "localized" i.e. Φ is localizing but not smoothing, e.g. :

$$\mathbf{E}(\xi(t,x)\bar{\xi}(s,y)) = k(x)\delta_{x-y}\delta_{t-s}$$

with $k \in L^2(\mathbf{R}^d)$ i.e. $\Phi u(x) = k(x)u(x)$, should take $s < -d/2$; however the deterministic NLS equation is ill posed in H^s, $s < 0$ (Kenig, Ponce & Vega, 2001)

Theorem 1 (local existence of solutions) :

- Assume $0 \le \sigma \le 2/d$, and $u_0 \in L^2(\mathbf{R}^d)$; assume Φ is a Hilbert-Schmidt operator in $L^2(\mathbf{R}^d)$. Then there is a random (stopping) time $T(u_0)$ and a unique adapted solution u which has paths a.s. in $C([0, T(u_0)); L^2) \cap L^r(0, T(u_0); L^{2\sigma+2})$ where $\frac{2}{r} = d(\frac{1}{2} - \frac{1}{2\sigma+2})$
- If $\sigma \ge 0$ for $d = 1, 2$ or $0 \le \sigma \le 2/(d-2)$ for $d \ge 3$, $\Phi \in HS(L^2, H^1)$ and $u_0 \in H^1$ then the same result is true in the space $C([0, T); H^1) \cap L^r((0, T); W^{1,2\sigma+2}(\mathbf{R}^d))$ for some random time $T = T(u_0)$. Moreover,

$$T(u_0) = +\infty \text{ or } \lim_{t \to T(u_0)} |\nabla u(t)|_{L^2}^2 = +\infty, \ a.s.$$

Proof : We recall that $u = v + Z$, with

$$v(t) = e^{it\Delta}u_0 + i\lambda \int_0^t e^{i(t-s)\Delta}(|v+Z|^{2\sigma}(v+Z))(s)ds$$

Strategy for the deterministic equation : fixed point in the space

$$X_T = L^\infty(0,T;L^2(\mathbf{R}^d)) \cap L^r(0,T;L^{2\sigma+2}(\mathbf{R}^d))$$

with $\frac{2}{r} = d(\frac{1}{2} - \frac{1}{2\sigma+2})$ and with the use of "Strichartz" estimates.
Thus it is sufficient to prove that $Z \in X_T$ a.s. under the conditions of Theorem 1.
We have

$$\mathbf{E}(\|Z\|^2_{L^\infty(0,T;L^2))}) = \mathbf{E}(\sup_{t\leq T} \| \int_0^t e^{i(t-s)\Delta} dW(s)\|^2_{L^2})$$

$$" = \mathbf{E}(\sup_{t\leq T} \| \int_0^t e^{-is\Delta} dW(s)\|^2_{L^2})"$$

$$\leq 4\mathbf{E}(\| \int_0^T e^{-is\Delta} dW(s)\|^2_{L^2}) \leq 4T\|\Phi\|^2_{HS(L^2)}$$

(martingale inequality)

On the other hand, since Z is a Gaussian process,

$$\mathbf{E}\int_0^T |Z(s)|^r_{L^{2\sigma+2}} ds \le C\int_0^T \mathbf{E}(|Z(s)|^{2\sigma+2}_{L^{2\sigma+2}})^{r/(2\sigma+2)} ds$$
$$\le C(\int_0^T (\int_{\mathbf{R}^d} \mathbf{E}(|Z(s,x)|^2)^{\sigma+1} dx)^{r/(2\sigma+2)} ds$$

and by the Itô isometry again

$$\mathbf{E}(|Z(s,x)|^2) = \mathbf{E}|\sum_k \int_0^s e^{i(s-\tau)\Delta}(\Phi e_k)(x) dB_k(\tau)|^2$$
$$= \sum_k \int_0^s |e^{i(s-\tau)\Delta}(\Phi e_k)(x)|^2 d\tau$$

Hence, by Minkowski inequality,

$$(\int_{\mathbf{R}^d} \mathbf{E}(|Z(s,x)|^2)^{\sigma+1} dx)^{1/\sigma+1} \le \sum_k \int_0^s |e^{i\tau\Delta}(\Phi e_k)(x)|^2_{L^{2\sigma+2}} ds$$
$$\le \sum_k \|e^{i.\Delta}(\Phi e_k)\|^2_{L^2(0,s;L^{2\sigma+2})}$$

We then use Strichartz estimates :

$e^{i.\Delta}$ maps $L^2(\mathbf{R}^d)$ into $L^r(0, T; L^{2\sigma+2})$ and since $r > 2$, we obtain

$$(\int_{\mathbf{R}^d} \mathbf{E}(|Z(s,x)|^2)^{\sigma+1} dx)^{1/\sigma+1} \leq C(T) \sum_k \|\Phi e_k\|^2_{L^2(\mathbf{R}^d)}$$
$$\leq C(T) \|\Phi\|^2_{HS(L^2)}$$

Finallly, integrating in time we obtain

$$\mathbf{E}\|Z\|^r_{L^r(0,T;L^{2\sigma+2})} \leq C(T)\|\Phi\|^r_{HS(L^2)}$$

which shows that $Z \in L^r(0, T; L^{2\sigma+2}(\mathbf{R}^d))$ a.s.

For the case $u_0 \in H^1$ and $\Phi \in HS(L^2, H^1)$, it is sufficient to note that $e^{it\Delta}$ commutes with space derivation.

Theorem 2 (globalization) :

Under the assumptions of Theorem 1, if either $\sigma < 2/d$ (subcritical) or $\lambda = -1$ (defocusing) then the solution is global i.e. $T(u_0) = +\infty$ a.s.

Remark : Same result as for the deterministic equation

Proof : The proof is based on the evolution of the L^2-norm $|u(t)|_{L^2}^2$ and of the Hamiltonian

$$H(u) = \frac{1}{2}\int_{\mathbf{R}^d} |\nabla u|^2 dx - \frac{\lambda}{2\sigma+2}\int_{\mathbf{R}^d} |u|^{2\sigma+2} dx$$

for the stochastic equation. Note that both are conserved for the deterministic equation.

The evolution is obtained thanks to the Itô Formula e.g. (here τ is a random stopping time)

$$
\begin{aligned}
|u(\tau)|_{L^2}^2 &= |u_0|_{L^2}^2 - 2\Im \sum_{m\in\mathbf{N}} \int_0^\tau \int_{\mathbf{R}^d} u(x)\overline{\Phi e_m}(x)\,dx\,dB_m(s) \\
&\quad + \tau\|\Phi\|_{HS(L^2)}^2
\end{aligned}
$$

from which we deduce

$$\mathbf{E}(|u(\tau)|_{L^2}^2) = |u_0|_{L^2}^2 + \tau\|\Phi\|_{HS(L^2)}^2$$

and we can actually prove (for $T > 0$ deterministic)

$$\mathbf{E}(\sup_{t\le T\wedge\tau} |u(t)|_{L^2}^2) \le C(T,\Phi,|u_0|_{L^2})$$

with the use of martingale (Doob) inequality. Then, since

$$T(u_0) = \lim_{R\to+\infty} \inf\{t, |u(t)|_{L^2}^2 > R\},$$

Markov inequality and the preceding bound shows that $\mathbf{P}(T(u_0) \le T) = 0$, if $\sigma < 2/d$ (L^2solutions).

For H^1 solutions, we use the evolution of the Hamiltonian (again, Itô Formula) :

$$\begin{aligned} dH(u(t)) &= -\Im \int_{\mathbf{R}^d} (\Delta \bar{u} + \lambda |u|^{2\sigma} \bar{u}) dW(t,x) dx \\ &\quad + \frac{1}{2} \|\nabla \Phi\|^2_{HS(L^2)} dt \\ &\quad - \frac{\lambda}{2} \sum_m \int_{\mathbf{R}^d} [|u|^{2\sigma} |\Phi e_m|^2 + 2\sigma |u|^{2\sigma-2} (\Re \bar{u} (\Phi e_m))^2] dx dt \end{aligned}$$

from which we deduce, if $\sigma < 2/d$ or $\lambda = -1$ that

$$\mathbf{E}(\sup_{t \le T \wedge \tau} |u(t)|^2_{H^1}) \le C(T, \Phi, \mathbf{E}(|u_0|^{2+\frac{4\sigma}{2-\sigma d}}), \mathbf{E}(H(u_0)))$$

which allows to conclude as before.

Remarks on the multiplicative case :

- Same kind of results as in the additive case, with additional technical restrictions on σ.
- More regularity is required on Φ. Need to work with Banach space valued Wiener processes (Hilbert-Schmidt condition replaced by a γ-radonifying condition on the operator Φ)
- Much more complicated proof : need the use of a cut-off (for nonlinear term) in $L^r(0,T;L^{2\sigma+2}(\mathbf{R}^d))$ norm and fixed point in $L^\rho(\Omega;X_T)$ with $X_T = L^\infty(0,T;L^2) \cap L^r(0,T;L^{2\sigma+2})$ on the truncated equation $\rightsquigarrow$ globalization argument is not obvious ; a.s. conservation of L^2 norm is essential

Blow-up for multiplicative NLS equations :

A. de Bouard & AD, Annals of Proba., 2005

Consider the focusing stochastic NLS equation

$$idu + (\Delta u + |u|^{2\sigma} u)dt = u \circ dW$$

with $W = \sum_k \Phi e_k B_k$, and $(B_k)_k$ family of independent real valued B.M. We recall that the equation is equivalent to

$$idu + (\Delta u + |u|^{2\sigma} u)dt = udW - \frac{i}{2} u F_\Phi dt$$

with $F_\Phi(x) = \sum_k (\Phi e_k)^2(x)$

Under adequate regularity assumptions on Φ, for a given $u_0 \in H^1(\mathbf{R}^d)$, there is a random (stopping) time $\tau^*(u_0) = \tau^*$ and a unique solutions u with paths a.s. in $C([0,\tau^*); H^1) \cap L^r(0,\tau^*; W^{1,2\sigma+2})$, for $0 < \sigma < 2/(d-2)$, if $d \leq 3$ and $0 < \sigma < 1/(d-1)$ if $d \geq 4$.

Moreover, if $\sigma < 2/d$, then the solution is global i.e. $\tau^*(u_0) = +\infty$ a.s.

Evolution of $M(u) = |u|^2_{L^2}$ and

$$H(u) = \frac{1}{2}\int_{\mathbf{R}^d} |\nabla u|^2 dx - \frac{1}{2\sigma+2}|u|^{2\sigma+2}dx$$

given by (again, Itô Formula ...)

$$M(u(\tau)) = M(u_0), a.s.$$

and

$$H(u(\tau)) = H(u_0) - \Im\int_{\mathbf{R}^d}\int_0^\tau \bar{u}\nabla u.\nabla dW(x,t)dx$$

$$+\tfrac{1}{2}\sum_{k\in\mathbf{N}}\int_0^\tau\int_{\mathbf{R}^d}|u|^2|\nabla\Phi e_k|^2 dxds$$

for any stopping time τ with $\tau < \tau^*(u_0)$ a.s.

The virial identity :

We can easily generalize the deterministic blow-up result by computing the evolution of the "virial" and momentum :

$$V(u) = \int_{\mathbf{R}^d} |x|^2 |u|^2 dx, \text{ and } G(u) = \Im \int_{\mathbf{R}^d} xu(x).\nabla u(x) dx$$

where we assume now that $u_0 \in \Sigma = \{v \in H^1,\ xv \in L^2\}$, and we can prove that in that case u has paths a.s. in $C([0,\tau^*);\Sigma)$. We find

$$\begin{aligned}
& V(u(\tau)) \\
&= V(u_0) + 4G(u_0)\tau + 8H(u_0)\tau^2 + 4\frac{(2-\sigma d)}{\sigma+1}\int_0^\tau \int_0^s |u(s_1)|_{L^{2\sigma+2}}^{2\sigma+2} ds_1 ds \\
&\quad +8\int_0^\tau \int_0^s \int_0^{s_1} \int_{\mathbf{R}^d} |u(s_2,x)|^2 f_\Phi^1(x) dx\, ds_2 ds_1 ds \\
&\quad +4\sum_k \int_0^\tau \int_0^s \int_{\mathbf{R}^d} |u(s_1,x)|^2 x.\nabla(\Phi e_k)(x) dx dB_k(s_1) ds \\
&\quad -16\Im \sum_k \int_0^\tau \int_0^s \int_0^{s_1} \int_{\mathbf{R}^d} \bar{u}(s_2,x)\nabla u(s_2,x).\nabla(\Phi e_k)(x) dx dB_k(s_2) ds_1 ds
\end{aligned}$$

The virial identity :

We can easily generalize the deterministic blow-up result by computing the evolution of the "virial" and momentum :

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where we assume now that $u_0 \in \Sigma = \{v \in H^1, \ xv \in L^2\}$, and we can prove that in that case u has paths a.s. in $C([0,\tau^*);\Sigma)$. This implies

$$\begin{aligned} & V(u(\tau)) \\ & \leq V(u_0) + 4G(u_0)\tau + 8H(u_0)\tau^2 + \frac{8}{3} m_\phi M(u_0)\tau^3 \\ & \quad + \textit{stochastic terms} \end{aligned}$$

We can actually use the above horrible identity to prove that if the initial state has sufficiently negative energy, then the corresponding solution blows up with positive probability in the supercritical case

Theorem : Assume $\sigma \geq 2/d$ (and regularity assumptions on Φ) ; Let $u_0 \in \Sigma$ be such that for some $\bar{t} > 0$,

$$(*) \quad V(u_0) + 4G(u_0)\bar{t} + 8H(u_0)\bar{t}^2 + \tfrac{4}{3}\bar{t}^3 m_\Phi M(u_0) < 0$$

(m_Φ is a constant depending on Φ) then

$$\mathbf{P}(\tau^*(u_0) \leq \bar{t}) > 0$$

Proof : Assuming $\bar{t} < \tau^*(u_0)$ a.s. an taking the expectation in the preceding equality, at $\tau = \bar{t}$, we find (formally) :

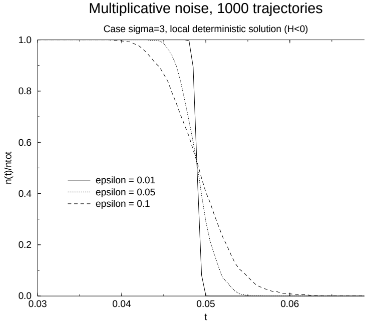
$$\mathbf{E}(V(u(\bar{t}))) \leq V(u_0) + 4G(u_0)\bar{t} + 8H(u_0)\bar{t}^2 + \frac{4}{3}\bar{t}^3 m_\Phi M(u_0) < 0$$

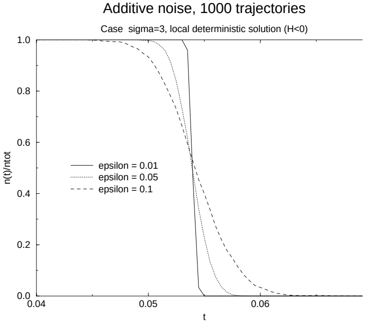
$\rightsquigarrow$ contradiction.

Numerical simulations of blow up probabilities

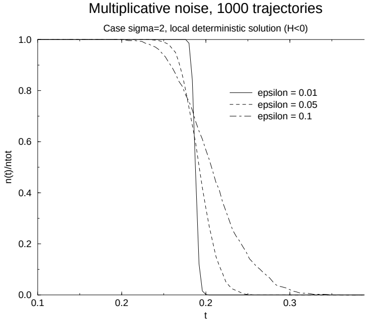
AD & L. Di Menza, 2002

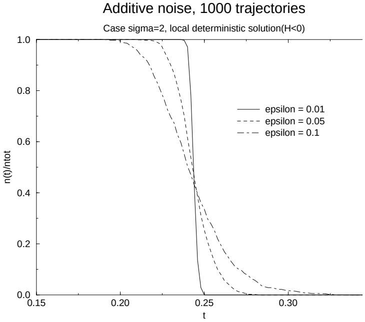
$$\sigma = 3$$



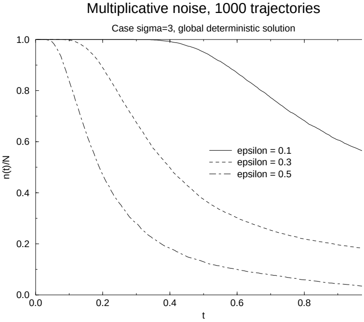


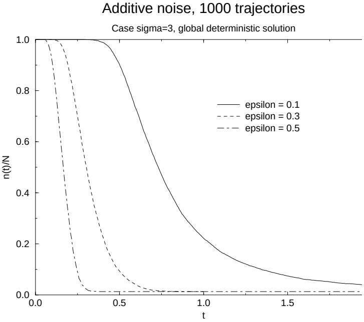
$$\sigma = 2$$


Multiplicative noise, 1000 trajectories
Case sigma=2, local deterministic solution (H<0)
epsilon = 0.01
epsilon = 0.05
epsilon = 0.1
n(t)/ntot
1.0
0.8
0.6
0.4
0.2
0.0
0.1
0.2
0.2
0.3
t


Additive noise, 1000 trajectories
Case sigma=2, local deterministic solution(H<0)
epsilon = 0.01
epsilon = 0.05
epsilon = 0.1
n(t)/ntot
1.0
0.8
0.6
0.4
0.2
0.0
0.15
0.20
0.25
0.30
t

$\sigma = 3$ (global deterministic solution)





$\sigma = 2$ (global deterministic solution)