Introduction to stochastic equations and some links with control theory

A. Debussche

École Normale Supérieure de Rennes

Mathematical Control in Trieste, SISSA, December 2nd-6th 2013

1 - Reminder

Brownian motion

Brownian motion (standard, nomalized)

$B = (B_t)_{t \geq 0}$ family of real random variables on $(\Omega, \mathcal{F}, \mathbf{P})$ such that

- $B(0) = 0$
- for a.e. $\omega \in \Omega$, $B(., \omega)$ is continuous in time
- any family of increments $(B(t_1) - B(t_0), ..., B(t_n) - B(t_{n-1})) = (X_1, ..., X_n)$ is an independent family (which is equivalent to $\mu_{(X_1,..,X_n)} = \mu_{X_1} \otimes ... \otimes \mu_{X_n}$)

This implies that $B(t)_{t \geq 0}$ is a Gaussian process and $B(t) - B(s)$ is centered with variance $|t - s|$
$\rightsquigarrow$ B is not more than $1/2$ regular.

One dimensional Itô integral

We want to give a meaning to : $\frac{dx}{dt} = f(x) + g(x)\dot{B}(t)$

$\rightsquigarrow$ we need to define " $\int_0^t g(x)\frac{dB}{dt}dt$" $= \int_0^t g(x)dB$ which is not a Stieljes integral.

Let $g \in C([0,T]; L^2(\Omega))$ such that $g(t)$ is $\mathcal{F}_t$-measurable for $t \in [0,T]$. We set

$$\int_0^T g(s)dB(s) = \lim_{|\sigma|\to 0} \sum_{i=0}^{n-1} g(t_i)(B(t_{i+1}) - B(t_i))$$

We note that :

- the expectation of the right hand side is equal to zero
- the variance of the right hand side is equal to

$$\sum_{i=0}^{n-1} \mathbf{E}(g^2(t_i))(t_{i+1} - t_i) = \int_0^T \mathbf{E}(\bar{g}^2(s))ds$$

where $\bar{g}$ is the piecewise constant process on the subdivision.

It follows : The Itô integral is defined by extending the isometry to "adapted" processes in $C([0,T];L^2(\Omega))$ (and actually for more general processes) with

- $\mathbf{E}(\int_0^T g(s)dB(s)) = 0$
- $\mathbf{E}\left((\int_0^T g(s)dB(s))^2\right) = \int_0^T \mathbf{E}(g^2(s))ds$

Notation : the equality $dX = fdt + gdB$ means that for any $t \in [0,T]$,

$$X(t) = X(0) + \int_0^t f(s)ds + \int_0^t g(s)dB(s)$$

1-D Itô formula : if $F \in C_b^2(\mathbf{R})$, then

$$dF(X) = F'(X)dX + \frac{1}{2}F''(X)g^2dt$$

Multi-dimensional case : B is a Brownian motion with values in $\mathbf{R}^n$ iff $B = (B_1, ..., B_n)$ with (B_i) family of independent B.M.

Infinite dimensional Wiener process :

Let

- H be a separable Hilbert space (typically, $H = L^2(O)$ or $H = L^2(\mathbf{R}^d)$)
- $(e_k)_{k\in\mathbf{N}}$ a complete orthonormal system in H
- $(B_k)_{k\in\mathbf{N}}$ a family of independent B.M.

then $W(t) = \sum_{k=0}^{\infty} e_k B_k$ is a cylindrical Wiener process on H.

Let $\Phi : H \to H$ be a bounded linear operator (smoothing) and let

$$\tilde{W}(t) = \sum_{j=0}^{\infty} \Phi e_j B_j = \Phi W$$

The series converges in $L^2(\Omega; H)$ iff Φ is Hilbert-Schmidt in H.

Infinite dimensional stochastic integral :

Let $\Psi \in L^2(0,T;\mathcal{L}(H,K))$ where H and K are separable Hilbert spaces. Let W be a cylindrical Wiener process on H.
Assume that Ψ is adapted (predictible) and

$$\mathbf{E}\int_0^t \|\Psi(s)\|^2_{HS(H,K)}ds < +\infty$$

Then if $(f_j)_{j\in\mathbf{N}}$ is a c.o.s. in K,

$$\int_0^t \Psi(s)dW(s) = \sum_{j,k}\int_0^t (\Psi(s)e_k, f_j)_K dB_k(s) f_j$$

defines a a.s. continuous process with values in K. Moreover

- $\mathbf{E}(\int_0^t \Psi(s)dW(s)) = 0$
- $\mathbf{E}(\|\int_0^t \Psi(s)dW(s)\|^2_K) = \mathbf{E}\int_0^T \|\Psi(s)\|^2_{HS(H,K)}ds$
- $\int_0^t \Psi(s)dW(s)$ is adapted

Itô formula :

Notation :

$$du = f(t)dt + \Psi(t)dW(t)$$

where $f(t)$ is K-valued, $\Psi(t)$ is $HS(H,K)$-valued.

Assume $K = H$ (for simplicity) and let F be a C_b^2 functional from H into $\mathbf{R}$; then for u as above,

$$dF(u) = (F'(u), du) + \frac{1}{2} Tr(F''(u)\Psi\Psi^*)dt$$

which means

$$\begin{aligned} F(u(t)) &= F(u_0) + \int_0^t (F'(u(s)), f(s))ds \\ &+ \int_0^t (F'(u(s)), \Psi(s)dW(s)) \\ &+ \tfrac{1}{2} \sum_l \int_0^t (F''(u(s))\Psi(s)e_l, \Psi(s)e_l)ds \end{aligned}$$

Remark on the Stratonovich integral :

One may define a stratonovich integral as (assuming $K = H$)

$$\int_0^t \Psi(s) \circ dW(s) = \lim \sum_l \psi(\frac{t_l + t_{l+1}}{2})(W(t_{l+1}) - W(t_l))$$

In the case $du = f(u)dt + \Psi(u) \circ dW$ we may rewrite in Itô form :

$$du = (f(u) + G_\Psi(u))dt + \Psi(u)dW$$

with

$$G_\Psi(u) = \frac{1}{2} \sum_l \Psi'(u)(\Psi(u)e_l)e_l \in H$$

The stochastic heat equation :

$$du = Audt + dW$$

where $A = -\Delta$ with boundary conditions and W a cylindrical Wiener process in $L^2(\mathcal{O})$ an open set in $\mathbf{R}^d$.
The solution is given by

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)}dW(s)$$

It is in $L^2(\mathcal{O})$ if $\int_0^t \|e^{A(t-s)}\|^2_{HS(L^2(\mathcal{O}))}ds < \infty.$ Therefore :

- $\mathcal{O}$ has to be a bounded open set.
- In this case

$$\begin{aligned}\int_0^t \|e^{A(t-s)}\|^2_{HS(L^2(\mathcal{O}))}ds &= \sum_k \int_0^t e^{-2\lambda_k(t-s)}ds \\ &= \sum_k \frac{1}{2\lambda_k}(1 - e^{-2\lambda_k t}) < \infty\end{aligned}$$

iff $d = 1$ since $\lambda_k \sim ck^{2/d}$.

2 - Control theory and properties of the stochastic NLS equations

2 - Control theory and properties of the stochastic NLS equations

2-1 Support theorems and application to blow up results

Girsanov formula :

Remark : Let X be a d-dimensional Gaussian random vector, with mean 0 and covariance matrix σ (invertible) , then

$$\mathbf{E}(\phi(X)) = \frac{1}{(2\pi)^{d/2}(\det\sigma)^{1/2}}\int_{\mathbf{R}^d}\phi(x)e^{-\frac{1}{2}(x,\sigma^{-1}x)}dx$$

and for $m \in \mathbf{R}^d$, by a simple computation,

$$\mathbf{E}(\phi(X)) = \frac{1}{(2\pi)^{d/2}(\det\sigma)^{1/2}}\int_{\mathbf{R}^d}\phi(y)e^{-\frac{1}{2}(y,\sigma^{-1}y)}dy$$
$$= \frac{1}{(2\pi)^{d/2}(\det\sigma)^{1/2}}\int_{\mathbf{R}^d}\phi(y-m)e^{(m,\sigma^{-1}y)-\frac{1}{2}(m,\sigma^{-1}m)}e^{-\frac{1}{2}(y,\sigma^{-1}y)}dy$$
$$= \mathbf{E}(\phi(X-m)e^{(m,\sigma^{-1}X)-\frac{1}{2}(m,\sigma^{-1}m)})$$
$$= \int_\Omega \phi(X-m)e^{(m,\sigma^{-1}X)-\frac{1}{2}(m,\sigma^{-1}m)}d\mathbf{P} = \int_\Omega \phi(\tilde{X})d\tilde{\mathbf{P}}$$

Hence $\tilde{X} = X - m$ is a centered Gaussian vector with variance σ on $(\Omega, \mathcal{F}, \tilde{\mathbf{P}})$ for the new probability measure

$$d\tilde{\mathbf{P}} = e^{(m,\sigma^{-1}X)-\frac{1}{2}(m,\sigma^{-1}m)}d\mathbf{P}$$

Girsanov theorem :

The property generalizes in the following way to stochastic integrals : Let $g \in L^2(0,T;H)$ a.s. be an adapted process satisfying the Novikov condition :

$$\mathbf{E}\left(\exp(\frac{1}{2}\int_0^T |g(s)|_H^2 ds)\right) < +\infty$$

then if W is a cylindrical Wiener process on H, for the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, then the process $\tilde{W}$ defined y

$$\tilde{W}(t) = W(t) - \int_0^t g(s)ds$$

is a cylindrical Wiener process on H for the probability space $(\Omega, \mathcal{F}, \tilde{\mathbf{P}})$ where

$$d\tilde{\mathbf{P}} = \exp\left(\int_0^T (g(s), dW(s)) - \frac{1}{2}\int_0^T |g(s)|_H^2 ds\right) d\mathbf{P}$$

This means that for any any functional of the Wiener process :

$$\mathbf{E}(\phi(W)) = \int_\Omega \phi(W)d\mathbf{P} = \tilde{\mathbf{E}}(\phi(\tilde{W})) = \int_\Omega \phi(\tilde{W})d\tilde{\mathbf{P}}.$$

Remark :

Note that by the Itô Formula, if the Novikov condition is satisfied and if $X(t) = \int_0^t (g(s), dW(s))$, then

$$d(e^X) = e^X dX + \frac{1}{2} e^X |g|_H^2 dt$$

so that

$$d(e^{X - \frac{1}{2}\int_0^t |g(s)|_H^2 ds}) = e^{X - \frac{1}{2}\int_0^t |g(s)|_H^2 ds} dX$$

and

$$e^{X(t) - \frac{1}{2}\int_0^t |g(s)|_H^2 ds} - 1 = \int_0^t e^{X(s)} (g(s), dW(s))$$

hence

$$\mathbf{E}\left(e^{\int_0^t (g(s), dW(s)) - \frac{1}{2}\int_0^t |g(s)|_H^2 ds} \right) = 1$$

. In other words :

$$\int_\Omega d\tilde{P} = \tilde{E}(1) = \mathbf{E}\left(e^{\int_0^t (g(s), dW(s)) - \frac{1}{2}\int_0^t |g(s)|_H^2 ds} \right) = 1$$

i.e. $\tilde{P}$ is a probability measure

Application of Girsanov formula :

We consider the equation :

$$du = (Au + f(u))dt + dW,\ u(0) = u_0$$

in $H = L^2(0,1)$. With $A = \Delta$ with Dirichlet boundary conditions and W a cylindrical Wiener process in $L^2(0,1)$ and $u_0 \in H$.

Recall that the heat equation :

$$dv = Avdt + dW,\ v(0) = u_0$$

has a unique solution $v(t) = e^{At}u_0 + \int_0^t e^{A(t-s)}dW(s)$. It lives in H and more precisely in $L^2(\Omega, C([0,T]; H))$.

Application of Girsanov formula :

We consider the equation :

$$du = (Au + f(u))dt + dW, \ u(0) = u_0$$

and $v \in L^2(\Omega, C([0,T]; H))$ solution of the heat equation :

$$dv = Avdt + dW, \ v(0) = u_0$$

Assume that $f : H \to H$ is bounded. Then :

$$\tilde{W}(t) = W(t) - \int_0^t f(v(s))ds$$

is a cylindrical Wiener process in $(\Omega, \mathcal{F}, \tilde{\mathbf{P}})$ where

$$d\tilde{\mathbf{P}} = \exp\left(\int_0^T (f(v(s)), dW(s)) - \frac{1}{2}\int_0^T |f(v(s))|_H^2 ds\right) d\mathbf{P}$$

and

$$dv = (Av + f(v))dt + d\tilde{W}$$

$\rightsquigarrow$ v is a solution of the nonlinear equation with a different Wiener process in a different probability space. We call that a martingale solution.

Application of Girsanov formula :

This is an information on the law :

$$du = (Au + f(u))dt + dW, \; u(0) = u_0$$

$$dv = Avdt + dW, \; v(0) = u_0$$

$$\tilde{W}(t) = W(t) - \int_0^t f(v(s))ds$$

$$dv = (Av + f(v))dt + d\tilde{W}$$

$$\rightsquigarrow \tilde{\mathbf{E}}\left(\phi(v)\right) = \mathbf{E}\left(\phi(v)e^{\left(\int_0^T (f(v(s)), dW(s)) - \frac{1}{2}\int_0^T |f(v(s))|_H^2 ds\right)}\right)$$

$\rightsquigarrow$ The law of the nonlinear equation is explicitely known and it is absolutely continuous with respect to the law of the Wiener process.

Support theorem for stochastic differential equations

Aim : Characterize the support of the measure $\mu_X = \mathbf{P} \circ X^{-1}$ on $C^\alpha([0,T])$ (with $\alpha < 1/2$) of a diffusion process X, given e.g. as the solution of the Itô stochastic differential equation

$$\begin{cases} dX = b(X)dt + \sigma(X)dB_t \\ X(0) = x \end{cases}$$

where B is a real valued B.M., $x \in \mathbf{R}$ is fixed and

$$\text{supp } \mu_X = \cap\{F, \text{ closed set of } C^\alpha([0,T]), \ \ \mu_X(F) = 1\}$$

(Stroock &Varadhan ; Ben Arous, Gradinaru & Ledoux ; Millet & Sanz-Sole)

We assume (for simplicity)

- σ of class C^2 on $\mathbf{R}$, with bounded first and second derivatives
- b globally lipschitz and bounded

Theorem : Under the above assumptions, for any $\alpha \in [0, 1/2)$, the support of the probability $\mu_X = \mathbf{P} \circ X^{-1}$ in $C^\alpha([0, T])$ is the closure $\mathcal{S}$ of the set $\{S(h),\ h \in L^2(0, T)\}$, where $S(h)(t) = X_t^h$ is the solution of

$$\begin{cases} \frac{dX_t^h}{dt} & = & b(X_t^h) - \frac{1}{2}\sigma'(X_t^h)\sigma(X_t^h) + \sigma(X_t^h)h(t) \\ X_t^h(0) & = & x \end{cases}$$

Remark :

- the set $\{S(h),\ h \in L^2(0, T)\}$ is called the "skeleton" of the diffusion process X
- the origin of the extra term in the drift is a Itô-Stratonovich correction. If instead consider the Stratonovich equation

$$dX = b(X)dt + \sigma(X) \circ dB$$

  the skeleton given by

$$\frac{dX_t^h}{dt} = b(X_t^h) + \sigma(X_t^h)h$$

Ideas of proof :

Consider an adapted linearly interpolated approximation B_t^n of the B.M. B_t, for $t \in [0, T]$, constructed as follows : for $\delta t = T/n$,

$$B_t^n = B_{t_{k-1}} + \frac{(t - t_k)}{\delta t}[B_{t_k} - B_{t_{k-1}}], \text{ for } t \in [t_k, t_{k+1}]$$

with $t_k = k(\delta t)$; then $\dot{B}_t^n$ is piecewise constant in time

• supp $\mu_X \subset \mathcal{S}$ (easy part) is obtained as follows :

First show that for any α with $0 \leq \alpha \leq 1/2$, $\mathbf{E}\|S(\dot{B}^n) - X\|_{C^\alpha}^2$ converges to 0 as $n \to \infty$. Write $X_n = \mathcal{S}(\dot{B}^n)$:

$$dX_n = b(X_n)dt - \frac{1}{2}\sigma'(X_n)\sigma(Xn) + \sigma(X_n)dB^n.$$

hence $\mu_{S(\dot{B}^n)} \to \mu_X$ as $n \to \infty$.

On the other hand,

$$\left\{S(\dot{B}^n)(\omega),\ \omega \in \Omega\right\} \subset \mathcal{S}, \text{ for any } \ n \in \mathbf{N}$$

so that $\mu_{S(\dot{B}^n)}(\mathcal{S}) = 1$ and hence $\mu_x(\mathcal{S}) = 1$

- $\mathcal{S} \subset \text{supp}\, \mu_X$ (more difficult) : We need to prove that

$$\mathbf{P}(\|X - S(h)\|_{C^\alpha([0,T])} < \varepsilon) = \mu_x(B(S(h), \varepsilon)) > 0$$

for any $\varepsilon > 0$ and for all $h \in L^2(0, T)$

Fix $\varepsilon > 0$ and $h \in L^2(0, T)$; we first prove that the solution Y_n of

$$\begin{cases} dY_n &= b(Y_n)dt + \sigma(Y_n)dB_t - \sigma(Y_n)\dot{B}^n dt + \sigma(Y_n)hdt \\ Y_n(0) &= x \end{cases}$$

converges to $S(h)$ in mean square in $C^\alpha([0, T])$. It follows that $Y_n \to S(h)$ in probability in $C^\alpha([0, T])$, and hence

$$\lim_{n\to\infty} \mathbf{P}(\|Y_n - S(h)\|_{C^\alpha} < \varepsilon) = 1$$

And, for n large enough,

$$\mathbf{P}(\|Y_n - S(h)\|_{C^\alpha} < \varepsilon) > 0$$

$$\begin{cases} dY_n &= b(Y_n)dt + \sigma(Y_n)dB_t - \sigma(Y_n)\dot{B}^n dt + \sigma(Y_n)hdt \\ Y_n(0) &= x \end{cases}$$

Now, by Girsanov Theorem,

$$\tilde{B}_n(t) = B(t) - \int_0^t \dot{B}^n(s)ds + \int_0^t h(s)ds$$

is a B.M. for a new probability $\mathbf{P}_n$ such that

$$d\mathbf{P} = e^{\int_0^T (\dot{B}_n(s)-h(s))d\tilde{B}_n - \frac{1}{2}\int_0^T |\dot{B}^n(s)-h(s)|^2 ds} d\mathbf{P}_n$$

(we can check that the Novikov condition is satisfied) and Y_n satisfies the equation

$$\begin{cases} dY_n &= b(Y_n)dt + \sigma(Y_n)d\tilde{B}_n \\ Y_n(0) &= x \end{cases}$$

i.e. $\mu_{Y_n} := \mathbf{P}_n \circ Y_n^{-1} = \mathbf{P} \circ X^{-1}$; it follows, since $\mathbf{P} \ll \mathbf{P}_n$, that

$$0 < \mathbf{P}_n(\|Y_n - S(h)\|_{C^\alpha} < \varepsilon) = \mathbf{P}(\|X - S(h)\|_{C^\alpha} < \varepsilon)$$

Blow-up for multiplicative NLS equations :

A. de Bouard & AD, Annals of Proba., 2005

Consider the focusing stochastic NLS equation

$$idu + (\Delta u + |u|^{2\sigma} u)dt = u \circ dW$$

with $W = \sum_k \Phi e_k B_k$, and $(B_k)_k$ family of independent real valued B.M. We recall that the equation is equivalent to

$$idu + (\Delta u + |u|^{2\sigma} u)dt = udW - \frac{i}{2} uF_\Phi dt$$

with $F_\Phi(x) = \sum_k (\Phi e_k)^2(x)$

Under adequate regularity assumptions on Φ, for a given $u_0 \in H^1(\mathbf{R}^d)$, there is a random (stopping) time $\tau^*(u_0) = \tau^*$ and a unique solutions u with paths a.s. in $C([0,\tau^*); H^1) \cap L^r(0,\tau^*; W^{1,2\sigma+2})$, for $0 < \sigma < 2/(d-2)$, if $d \leq 3$ and $0 < \sigma < 1/(d-1)$ if $d \geq 4$.

Moreover, if $\sigma < 2/d$, then the solution is global i.e. $\tau^*(u_0) = +\infty$ a.s.

Evolution of $M(u) = |u|_{L^2}^2$ and

$$H(u) = \frac{1}{2}\int_{\mathbf{R}^d} |\nabla u|^2 dx - \frac{1}{2\sigma+2}|u|^{2\sigma+2}dx$$

given by (again, Itô Formula ...)

$$M(u(\tau)) = M(u_0), a.s.$$

and

$$H(u(\tau)) \quad = \quad H(u_0) - \Im \int_{\mathbf{R}^d}\int_0^\tau \bar{u}\nabla u.\nabla dW(x,t)dx$$
$$+\tfrac{1}{2}\sum_{k\in\mathbf{N}}\int_0^\tau\int_{\mathbf{R}^d}|u|^2|\nabla\Phi e_k|^2 dxds$$

for any stopping time τ with $\tau < \tau^*(u_0)$ a.s.

The virial identity :

We can easily generalize the deterministic blow-up result by computing the evolution of the "virial" and momentum :

$$V(u) = \int_{\mathbf{R}^d} |x|^2 |u|^2 dx, \text{ and } G(u) = \Im \int_{\mathbf{R}^d} xu(x).\nabla u(x) dx$$

where we assume now that $u_0 \in \Sigma = \{v \in H^1,\ xv \in L^2\}$, and we can prove that in that case u has paths a.s. in $C([0,\tau^*);\Sigma)$. We find

$$\begin{aligned}
&V(u(\tau)) \\
&= V(u_0) + 4G(u_0)\tau + 8H(u_0)\tau^2 + 4\frac{(2-\sigma d)}{\sigma+1}\int_0^\tau \int_0^s |u(s_1)|_{L^{2\sigma+2}}^{2\sigma+2} ds_1 ds \\
&\quad +8\int_0^\tau \int_0^s \int_0^{s_1} \int_{\mathbf{R}^d} |u(s_2,x)|^2 f_\Phi^1(x) dx\, ds_2 ds_1 ds \\
&\quad +4\sum_k \int_0^\tau \int_0^s \int_{\mathbf{R}^d} |u(s_1,x)|^2 x.\nabla(\Phi e_k)(x) dx dB_k(s_1) ds \\
&\quad -16\Im \sum_k \int_0^\tau \int_0^s \int_0^{s_1} \int_{\mathbf{R}^d} \bar{u}(s_2,x)\nabla u(s_2,x).\nabla(\Phi e_k)(x) dx dB_k(s_2) ds_1 ds
\end{aligned}$$

The virial identity :

We can easily generalize the deterministic blow-up result by computing the evolution of the "virial" and momentum :

$$V(u) = \int_{\mathbf{R}^d} |x|^2 |u|^2 dx, \text{ and } G(u) = \Im \int_{\mathbf{R}^d} x u(x).\nabla u(x) dx$$

where we assume now that $u_0 \in \Sigma = \{v \in H^1,\ xv \in L^2\}$, and we can prove that in that case u has paths a.s. in $C([0,\tau^*);\Sigma)$. This implies

$$\begin{aligned} &V(u(\tau)) \\ &\leq V(u_0) + 4G(u_0)\tau + 8H(u_0)\tau^2 + \frac{8}{3} m_\phi M(u_0)\tau^3 \\ &\quad + \textit{stochastic terms} \end{aligned}$$

We can actually use the above horrible identity to prove that if the initial state has sufficiently negative energy, then the corresponding solution blows up with positive probability in the supercritical case

Theorem : Assume $\sigma \geq 2/d$ (and regularity assumptions on Φ) ; Let $u_0 \in \Sigma$ be such that for some $\bar{t} > 0$,

$$(*) \quad V(u_0) + 4G(u_0)\bar{t} + 8H(u_0)\bar{t}^2 + \tfrac{4}{3}\bar{t}^3 m_\Phi M(u_0) < 0$$

(m_Φ is a constant depending on Φ) then

$$\mathbf{P}(\tau^*(u_0) \leq \bar{t}) > 0$$

Proof : Assuming $\bar{t} < \tau^*(u_0)$ a.s. an taking the expectation in the preceding equality, at $\tau = \bar{t}$, we find (formally) :

$$\mathbf{E}(V(u(\bar{t}))) \leq V(u_0) + 4G(u_0)\bar{t} + 8H(u_0)\bar{t}^2 + \frac{4}{3}\bar{t}^3 m_\Phi M(u_0) < 0$$

$\rightsquigarrow$ contradiction.

Remark : The same is true if u_0 is random ($\mathcal{F}_0$-measurable) and if $(*)$ is satisfied only on a $\mathcal{F}_0$-measurable set $\Omega_0 \subset \Omega$ with $\mathbf{P}(\Omega_0) > 0$

If $\bar{M}$ and $\bar{H}$ are positive constants, we denote

$$\mathcal{V}_{\bar{M},\bar{H}} = \left\{ v \in \Sigma,\ V(v) < \bar{M},\ G(v) < \bar{M},\ |v|_{L^2}^2 < \bar{M},\ H(v) < -\bar{H} \right\}$$

Corollary : Let u_0, Φ, σ as in the existence theorem and let $\sigma > 2/d$; assume $u_0 \in \Sigma$, a.s., then for any $\bar{M} > 0$ and $\bar{t} > 0$, there exist a constant $\bar{H}(\bar{t}, \bar{M}) > 0$ such that

$$\mathbf{P}(u_0 \in \mathcal{V}_{\bar{M},\bar{H}}) > 0 \Rightarrow \mathbf{P}(\tau^*(u_0) < \bar{t}) > 0$$

Proof : Indeed, given $\bar{M} > 0$, $\bar{t} > 0$, it is sufficient to consider $\bar{H} > 0$ large enough so that

$$\bar{M} + 4\bar{t}\bar{M} - 8\bar{t}^2\bar{H} + \frac{4}{3}\bar{t}^3 m_\Phi \bar{M} < 0;$$

then applying the above argument with

$$\Omega_0 = \left\{\omega \in \Omega,\ u_0(.,\omega) \in \mathcal{V}_{\bar{M},\bar{H}}\right\}$$

we obtain that $(*)$ is satisfied on Ω_0 (with $\mathbf{P}(\Omega_0) > 0$ from the assumptions) $\rightsquigarrow$ $\mathbf{P}(\tau^*(u_0) < \bar{t}) > 0$ $\square$

Now, if the solution is "non degenerate" (i.e. the support of the measure is the whole space) we may expect that given any u_0, the solution will visit $\mathcal{V}_{\bar{M},\bar{H}}$ with a positive probability at some time t_1, and hence the solution will blow up before $t_1 + \bar{t}$. The support theorem suggests that non degeneracy is a controllability problem.

The control problem

We actually need a weaker result, which is the following :

Proposition : Assume $\sigma > 2/d$ and let $u_0 \in \Sigma$ with $u_0 \neq 0$ and let $T > 0$ fixed ; we set

$$\bar{M} = \max(M(u_0),\ V(u_0) + 4T|G(u_0)|, |G(u_0)|)$$

Then for any $\bar{H} > 0$, we can find $t_1 < T$ and a real valued potential $f \in L^s(0, t_1; W^{1,p})$ (for some $s > 1$ and some $p < (\sigma+1)/\sigma$) such that the solution U of

$$\begin{cases} i\dfrac{dU}{dt} + \Delta U + |U|^{2\sigma} U = f\, U \\ U(0) = u_0 \end{cases}$$

exist on $[0, t_1)$ and satisfies $U(t_1) \in \mathcal{V}_{\bar{M},\bar{H}}$

Remark : Conditions $\sigma > 2/d$ and $u_0 \neq 0$ are essential. For $\sigma = 2/d$ we have a weaker result.

Proof : In order to construct the control f, we consider for $\tilde{\sigma}$ such that $2/d < \tilde{\sigma} < \sigma$:

$$\begin{cases} i\dfrac{dU}{dt} + (\Delta U + \lambda |U|^{2\tilde{\sigma}} U) = 0 \\ U(0) = u_0 \end{cases}$$

The constant $\lambda > 0$ is chosen large enough, in order that the Hamiltonian :

$$H_{\tilde{\sigma},\lambda}(U(t)) = H_{\tilde{\sigma},\lambda}(u_0) = \frac{1}{2}\int_{\mathbf{R}^d} |\nabla u_0|^2 dx - \frac{\lambda}{2\tilde{\sigma}+2}\int_{\mathbf{R}^d} |u_0|^{2\tilde{\sigma}+2} dx$$

(conserved for the above equation) satisfies

$$V(u_0) + 4TG(u_0) + 8T^2 H_{\tilde{\sigma},\lambda}(u_0) < 0$$

Then, deterministic theory ensures existence of a unique local solution $U(t) \in C([0, T^*), H^1(\mathbf{R}^d))$ which blows up at T^*, with $T^* < T$.

In particular, $\lim_{t\to T^*} |U(t)|_{H^1(\mathbf{R}^d)} = +\infty$ and since $H_{\tilde\sigma,\lambda}(U(t))$ is conserved, $\lim_{t\to T^*} |U(t)|_{L^{2\tilde\sigma+2}} = +\infty$; now, since the L^2 norm of U is conserved, Hölder inequality implies

$$H(U(t)) \leq -\delta |U(t)|^{2\tilde\sigma+2}_{L^{2\tilde\sigma+2}} + H_{\tilde\sigma,\lambda}(U(t)) + C(\lambda,\sigma,\tilde\sigma)|u_0|^2_{L^2}$$

and hence

$$\lim_{t\to T^*} H(U(t)) = -\infty$$

with $H = H_{\sigma,1}$ the Hamiltonian of the original equation. Hence we may choose $t_1 < T^*$ such that $H(U(t_1)) < -\bar H$; moreover, given the definition of $\bar M$,

$$V(U(t_1)) \leq V(u_0) + 4t_1|G(u_0)| < \bar M$$

and

$$G(U(t_1)) \leq G(u_0) + 4t_1 H_{\tilde\sigma,\lambda}(u_0) < G(u_0) < \bar M$$

hence $U(t_1) \in \mathcal{V}_{\bar M,\bar H}$. Setting $f(t) = \lambda|U(t)|^{2\tilde\sigma} - |U(t)|^{2\sigma}$, we have solved the problem.

The support property and refined blow up result

We now go back to our stochastic equation

$$idu + (\Delta u + |u|^{2\sigma} u)dt = u \circ dW$$

In order to apply the preceding argument, we need prove that the solution u visits any neighborhood of $U(t_1)$ with a positive probability, where $U(t_1) \in \mathcal{V}_{\bar{M},\bar{H}}$ is as above.

This requires some additional regularity on u_0 and a non degeneracy condition on the noise.

Proposition : Assume $u_0 \in \Sigma^2$, Φ regular and $N((\Phi|_{L^2(\mathbf{R}^d,\mathbf{R})})^*) = \{0\}$; Let $T > 0$, $\bar{H} > 0$ and $t_1 < T$, U, f as above. Then for any neighborhood $\mathcal{V}$ of $U(t_1)$ in Σ, the solution u satisfies

$$\mathbf{P}(\tau^*(u_0) > t_1 \text{ and } u(t_1) \in \mathcal{V}) > 0$$

Ideas of proof : Inspired by proof of support theorem.

Since $N((\Phi|_{L^2(\mathbf{R}^d,\mathbf{R})})^*) = \{0\}$, there exists a sequence g_n with $\lim_{n\to\infty} \Phi g_n = f$ in $C([0,T);H^2)$.

Then consider the cylindrical Wiener process $W_c(t) = \sum_k e_k B_k(t) = \Phi^{-1}W(t)$ and consider a finite dimensional adapted linear interpolation $\dot{W}_{c,n}$ of W_c.

Let u^n be the solution of

$$idu^n + (\Delta u^n + |u^n|^{2\sigma}u^n + f_n u^n)dt = u^n \circ dW - u^n \dot{W}_n dt$$

with $W_n = \Phi W_n$.

- We prove that u^n converges to U in probability as $n \to +\infty$; hence for n large enough

$$\mathbf{P}(\tau_n^*(u_0) > t_1 \text{ and } u^n(t_1) \in \mathcal{V}) > 0$$

- On the other hand, Girsanov theorem implies

$$W_n(t) = W(t) - \int_0^t (\dot{W}_n(s) - \Phi g_n(s))ds$$

is a Wiener process with covariance operator $\Phi\Phi^*$, for the new probabiilty measure $d\mathbf{P}_n = D_n d\mathbf{P}$, with

$$D_n = e^{\int_0^T (\Phi^{-1}\dot{W}_n(s) - g_n(s), \Phi^{-1}dW(s)) - \frac{1}{2}\int_0^T |\Phi^{-1}\dot{W}_n(s) - g_n(s)|^2_{L^2} ds}$$

and u^n is then solution of

$$idu^n + (\Delta u^n + |u^n|^{2\sigma} u^n)dt = u^n \circ dW_n$$

$$\rightsquigarrow \mathbf{P}(\tau^*(u_0) > t_1, \ \ u(t_1) \in \mathcal{V}) = \mathbf{P}_n(\tau_n^*(u_0) > t_1, \ \ u_n(t_1) \in \mathcal{V}) > 0.$$

Finally, for $u_0 \in \Sigma^2$, $u_0 \neq 0$ and $t > 0$, fix $T > 0$ and $\bar{t} > 0$ such that $T + \bar{t} \leq t$ and fix $\bar{H}, \bar{M} > 0$ as above ; we obtain that for some $t_1 < T$,

$$\mathbf{P}(\{\omega \in \Omega, \ \tau^*(u_0) > t_1, \text{ and } u(t_1) \in \mathcal{V}_{\bar{M},\bar{H}}\}) > 0$$

Markov property and the previous corollary imply that the solution u of the stochastic equation blows up before $t_1 + \bar{t}$ with a positive probability. We thus have proved :

Theorem : Assume $\sigma > 2/d$, Φ regular and $N((\Phi|_{L^2(\mathbf{R}^d,\mathbf{R})})^*) = \{0\}$; then, for any $u_0 \in \Sigma^2$ with $u_0 \neq 0$, and for any $t \geq 0$,

$$\mathbf{P}(\tau^*(u_0) \leq t) > 0$$

i.e. the solution blows up before time t with a positive probability

Remark on the additive case :

Same result true for the additive equation, the proof is actually simpler ($\sigma \geq 2/d$ allowed) :

- The support result may be replaced by the fact that the solution u is a continuous function of Z with

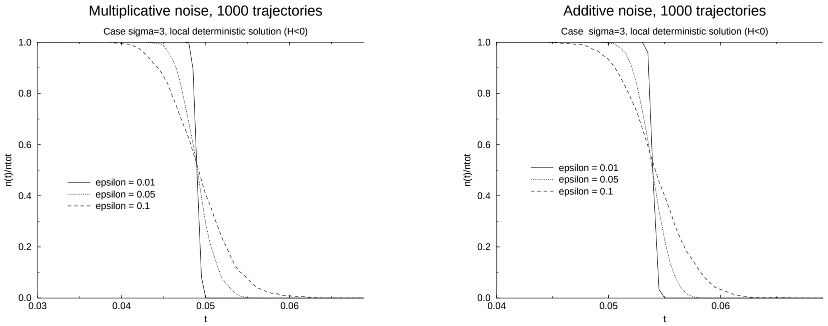
$$Z(t) = \int_0^t e^{i(t-s)\Delta} dW(s)$$

- The construction of the control is also easier (if W is complex valued) (linear control problem)
- We can also construct a control if W is real valued, but this requires more regularity on the data

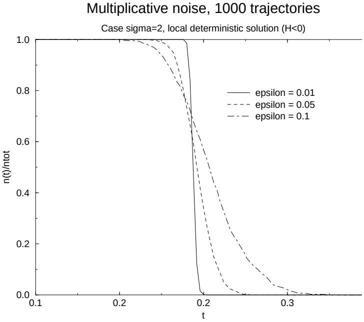
Numerical simulations of blow up probabilities

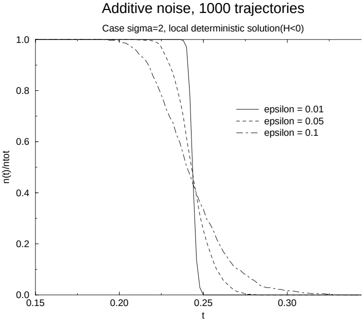
AD & L. Di Menza, 2002

$$\sigma = 3$$

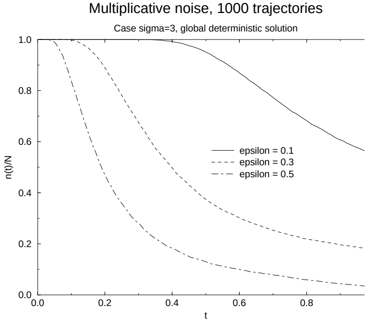


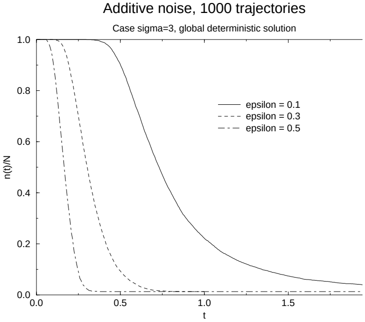
$\sigma = 2$


Multiplicative noise, 1000 trajectories
Case sigma=2, local deterministic solution (H<0)
1.0
0.8
0.6
0.4
0.2
0.0
n(t)/ntot
0.1
0.2
0.2
0.3
t
epsilon = 0.01
epsilon = 0.05
epsilon = 0.1

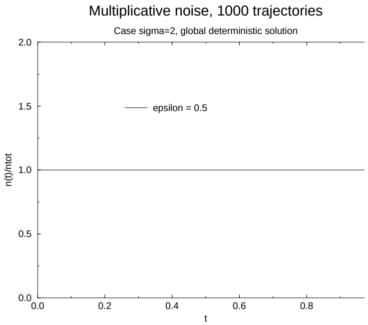

Additive noise, 1000 trajectories
Case sigma=2, local deterministic solution(H<0)
1.0
0.8
0.6
0.4
0.2
0.0
n(t)/ntot
0.15
0.20
0.25
0.30
t
epsilon = 0.01
epsilon = 0.05
epsilon = 0.1

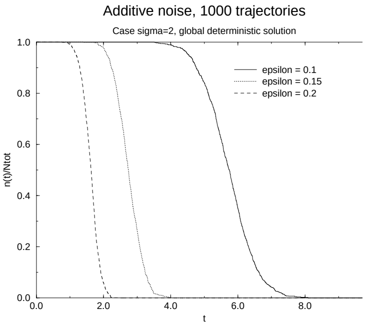
$\sigma = 3$ (global deterministic solution)





$\sigma = 2$ (global deterministic solution)





Open problems on support of solutions

- Bose-Einstein condensation :

$$idu + (\Delta u - |x|^2 u + |u|^{2\sigma} u)dt = |x|^2 u \circ dB_t$$

blow up results for some inital data if $\sigma \geq 2/d$; blow up with positive probability for any data : open pb (control pb)

- Propagation in optical fibers :

$$idu + \Delta u \circ dB_t + |u|^{2\sigma} udt = 0$$

Dispersion managed fibers ; global existence for $\sigma < d/2$ (or $\sigma \leq 2$ in 1-D) ; blow-up for $\sigma > d/2$: open pb

$$idU + (\partial_x^2 U + F(U))dt + \sum_{k=1}^{3} \sigma_k \partial_x U \circ dB_k(t) = 0$$

Polarisation mode dispersion ; $U(t,x) \in \mathbf{C}^2$, σ_k : Pauli matrices ; $F(U)$ cubic ; local exist. OK ; blow up : open pb

2 - Control theory and properties of the stochastic NLS equations

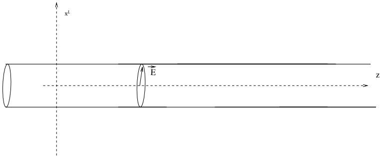
2 - 2 Large deviations and application to error transmission

Position of the problem

Additive NLS equations in 1-D : model for propagation of light in optical fibers (anomalous dispersion)

$$i\partial_z u + \partial_t^2 u + |u|^2 u = \varepsilon\xi(z,t)$$

u : complex envelope of electric field amplitude ; t : retarded time



noise due to amplification

assumption : distance between amplifiers $\ll$ fiber length ($\sim$ 1000 km) ; amplification compensates loss

We use mathematical notations ($z \to t$; $t \to x$)

Soliton propagation :

- the soliton solutions of the deterministic equation (localized pulses)

$$\Psi_A^v(t,x) = \sqrt{2}A\,\mathrm{sech}(A(x-x_0+2vt))\exp(i(v.x+v^2t-A^2t+\theta_0))$$

are coding for 0 and 1 (numerical information)

- At coordinate T (end of the line) a receiver records

$$\frac{1}{l}\int_{-l/2}^{l/2} |u(t,x)|^2 dx$$

with $u(0) = 0$ or $u(0) = \psi_A^v(0)$. If this quantity is above a certain threshold, it is decided that a "1" has been transmitted.

- Presence of noise $\rightsquigarrow$ signal distortion (soliton is no more a solution) with small probability ($\varepsilon l \ll 1$)

Computation of errors : two sources of errors :

- Error in signal transmission : e.g. $u(0) = \Psi_A(0)$ but no soliton has been detected at the end of the fiber ; estimator : mass

$$M(u(T)) = \|u(T)\|_{L^2}^2$$

- Error in signal detection (timing jitter) : a soliton has been transmitted, but shift in the velocity $\rightsquigarrow$ no detection estimator : center of mass

$$Y(u(T)) = \int_{\mathbf{R}} x|u(t,x)|^2 dx$$

Error rate $\sim 10^{-12}$ $\rightsquigarrow$ difficult to compute on the original equation (ISMC methods or genealogical systems ; see e.g. Del Moral & Garnier, 2005)

Results in the physics literature

- Gordon & Hauss, 1986 ; Drummond & Corney, 2001 : computations of the dependence in T of the variances (of $M(u)$ and $Y(u)$) $\rightsquigarrow$ deduce rate limitation via Gaussian approximation
- Falkovitch, Kolokolov, Lebedev, Mezentsev & Turitsyn, 2004 ; Derevyanko, Turitsyn & Yakushev, 2003 : computation of densities using ansatz on the solutions ($\rightsquigarrow$ finite dimension) + Fokker Planck equations
- Moore, Biondini & Kath, 2003 : computation of CDF using ansatz on the solution and ISMC methods

AD & E. Gautier, 2005 : Use of large deviation principle to prove rigorous bounds on the probabilities

Large deviation principle for additive NLS equation

We denote by u^{ε,u_0} the solution of the additive NLS equation

$$idu + (\partial_x^2 u + |u|^2 u)dt = \varepsilon dW(t,x)$$

with $W = \sum_k \Phi e_k B_k$; assume Φ is regular and $u_0 \in \Sigma$; then the measure $\mu^{\varepsilon,u_0} = \mathbf{P} \circ (u^{\varepsilon,u_0})^{-1}$ satisfies a large deviation principle in $C([0,T];\Sigma^{1/2})$, with (good) rate function

$$I^{u_0}(w) = \frac{1}{2} \inf_{\substack{h \in L^2(0,T;L^2_x) \\ S(u_0,h)(T) = w}} \|h\|^2_{L^2(0,T;L^2)}$$

where we denote by $S(u_0,h)$ the solution of

$$(NLSh) \qquad \begin{cases} i\dfrac{\partial u}{\partial t} + \partial_x^2 u + |u|^2 u = \Phi h(t,x) \\ u(0) = u_0 \end{cases}$$

Here,

$$\Sigma^{1/2} = \left\{ v \in H^1, \ \sqrt{|x|} v \in L^2 \right\}$$

This means : For any Borel set $B \subset C([0, T]; \Sigma^{1/2})$,

$$- \inf_{w \in \text{int}(B)} I^{u_0}(w) \leq \liminf_{\varepsilon \to 0} \varepsilon \log \mathbf{P}(u^{\varepsilon, u_0} \in B)$$

and

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbf{P}(u^{\varepsilon, u_0} \in B) \leq - \inf_{w \in \bar{B}} I^{u_0}(w)$$

Remark : As $\varepsilon \to 0$, the solution u of the stochastic equation converges to the solution of the deterministic equation. If B does not contain the deterministic trajectory, then $\mathbf{P}(u^{\varepsilon, u_0} \in B)$ is exponentially small for $\varepsilon \ll 1$; the LDP says that the exponential factor is given by the minimal "energy" $\|h\|^2_{L^2(0,T;L^2)}$ required so that the solution of the control pb $(NLSh)$ lives in B

Application to the tails of the mass

We apply the LDP with

- either $u_0 = 0$ and

$$B = \left\{ u \in C([0,T];\Sigma^{1/2}),\ M(u(T)) \geq R \right\}$$

(R large $\rightsquigarrow$ deviations from the zero solution)

- or $u_0 = \Psi_1^0$ (soliton with velocity $= 0$ and amplitude $= 1$) and

$$B = \left\{ u \in C([0,T];\Sigma^{1/2}),\ M(u(T)) \leq 4 - R \right\}$$

Note that $M(\Psi_1^0) = 4$ so that again if $4 - R \geq 1$, B contains deviations from Ψ_1^0

The aim is to find lower and upper bounds on $I^{u_0}(w)$ that imply upper and lower bounds on the probabilities, of the same order in the parameters

Note that :

- Need lower bounds on $\inf_{w\in\bar{B}} I^{u_0}(w)$ in order to obtain upper bounds on the probabilities. These are obtained thanks to energy estimates on the controled equation (which give the fact that $\|h\|_{L^2(0,T;L^2)}$ cannot be too small)
- Need upper bounds on $\inf_{w\in \mathrm{int}(B)} I^{u_0}(w)$ in order to obtain lower bounds on the probabilities. For that, use control that give solutions in the form of a modulated soliton, and try to optimize the parameter in order to obtain the best possible upper bound on $\inf_{w\in \mathrm{int}(B)} I^{u_0}(w)$ ($\rightsquigarrow$ control problem)

Upper bounds : Assume Φ is a Hilbert-Schmidt operator on $L^2(\mathbf{R})$; then

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbf{P}(M(u^{\varepsilon,0}(T)) \geq R) \leq -\frac{R}{8T\|\Phi\|^2_{\mathcal{L}(L^2)}}$$

and

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbf{P}(M(u^{\varepsilon,\Psi_1^0}(T)) \leq 4 - R) \leq -\frac{C_1(R)}{T\|\Phi\|^2_{\mathcal{L}(L^2)}}$$

Lower bounds : Now, Φ_n is a sequence of HS operators converging pointwize to Id ; then

$$\liminf_{n \to +\infty} \liminf_{\varepsilon \to 0} \varepsilon \log \mathbf{P}(M(u^{\varepsilon,0,n}(T)) \geq R) \geq -\frac{R(12+\pi^2)}{18T}$$

and

$$\liminf_{n \to +\infty} \liminf_{\varepsilon \to 0} \varepsilon \log \mathbf{P}(M(u^{\varepsilon,\Psi_1^0,n}(T)) \leq 4 - R) \geq -\frac{C_2(R)}{T}$$

with $u^{\varepsilon,u_0,n}$ the solution of the stochastic equation with Φ replaced by Φ_n

Applications to the tails of the arrival time

Again, apply the LDP with $u_0 = \Psi_A^0$, soliton with zero velocity and amplitude A. Consider $B = \left\{u \in C([0,T]; \Sigma^{1/2}),\ |Y(u(T))| \geq R\right\}$ for $R = O(1)$ and T large. Since $Y(\Psi_A^0) = 0$, the set B contains large deviations from the deterministic trajectory.

Upper bounds : Assume Φ Hilbert-Schmidt operator from L^2 into Σ ; then

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mathbf{P}(Y(u^{\varepsilon,\Psi_A^0}(T)) \geq R) \leq -\frac{C(R)}{T^3 A \|\Phi\|_{\mathcal{L}(L^2;\Sigma)}}$$

for $T \geq 1$, and $R/T \leq A$.

Lower bounds : Let Φ_n sequence of HS operators (from L^2 into Σ) converging pointwise to Id ; then

$$\liminf_{n \to +\infty} \liminf_{\varepsilon \to 0} \varepsilon \log \mathbf{P}(Y(u^{\varepsilon,\Psi_A^0,n}(T)) \geq R) \geq -\frac{\pi^2 R^2}{128\, T^3}$$

Comments on the results :

Results are coherent with those given in physics literature :

- Tails of the mass with 0 as initial state are not Gaussian, indistinguishable from exponential tails on a log scale
- Tails of the arrival time are indistinguishable from Gaussian tails on a log scale
- In terms of fiber length, tails of the mass given by $\exp(-\frac{C}{\varepsilon T})$ while tails of arrival time given by $\exp(-\frac{C}{\varepsilon T^3})$
  $\rightsquigarrow$ tails of arrival time are larger
  $\rightsquigarrow$ fluctuations of arrival time (timing jitter) is the main process impairing soliton optical communications
- Variance of arrival time $\sim AT^3$ also agrees with physical literature

Ideas of Proofs :

Tails of the mass, lower bounds, $u_0 = 0$:

Recall that soliton with 0 velocity is given by

$$\Psi_A(t,x) = \sqrt{2}A\,\mathrm{sech}(Ax)\exp(iA^2 t)$$

Assume $\Phi = Id$; we look for an explicit solution, for some $h \in L^2(0,T;L^2)$, to the controlled equation

$$(NLSh) \qquad i\frac{\partial u}{\partial t} + \partial_x^2 u + |u|^2 u = h$$

with $u(0) = 0$ and with $M(u(T)) \geq R$, of the form

$$S(h_A)(t,x) = u(t,x) = \sqrt{2}A(t)\,\mathrm{sech}(A(t)x)\exp(i\int_0^t A^2(s)ds)$$

with a real valued function $A(t)$, chosen to minimise $\|h_A\|^2_{L^2(0,T;L^2)}$

Plugging the above expression of $S(h_A)$ into the equation $(NLSh)$, one finds

$$\begin{aligned} h_A(t,x) &= i\left(\frac{A'(t)}{A(t)}\right)\Psi_A(t) \\ &\quad -i\sqrt{2}A'(t)A(t)x\frac{\sinh(A(t)x)}{\cosh^2(A(t)x)}\exp(i\int_0^t A^2(s)ds) \end{aligned}$$

and

$$\frac{1}{2}\|h_A\|^2_{L^2(0,T;L^2)} = \frac{(\pi^2+12)}{18}\int_0^T \frac{(A'(t))^2}{A(t)}dt = \int_0^T L(t,A(t),\dot{A}(t))dt$$

Hence, need to find a real valued function A minimizing the above integral, with the constraints $A(0)=0$ and $M(S(h_A)(T)) \geq R$; we may also compute, since $M(\Psi_A)=4A$:

$$M(S(h_A)(T)) = 4A(T).$$

The Euler-Lagrange equation :

$$\frac{\partial L}{\partial A} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{A}}\right] = 0$$

then gives

$$(A')^2 - 2A''A = 0$$

which implies $A''' = 0$ and $A(t) = \alpha(t-\beta)^2$; from the boundary constraints $A(0) = 0$ and $4A(T) \geq R$, we deduce

$$A(t) = \frac{R}{4}\left(\frac{t}{T}\right)^2 \text{ and } \frac{1}{2}\|h_A\|^2_{L^2(0,T;L^2)} = \frac{(12+\pi^2)}{18}\frac{R}{T}$$

If $\Phi \neq Id$, the above argument does not work (cannot find an expression of h_A, and hence of $\|h_A\|^2_{L^2(0,T;L^2)}$ that depends only on $A(t)$)

$\rightsquigarrow$ use of an approximation argument with $\lim_{n+\infty} \Phi_n = Id$

Tails of the mass, upper bounds, $u_0 = 0$:

Assume now Φ Hilbert-Schmidt with values in $H^1(\mathbf{R})$:

- Manipulations on the controlled equation

$$i\frac{\partial u}{\partial t} + \Delta u + |u|^2 u = \Phi h$$

allow to obtain (with $S(h)(0) = 0$) :

$$\|S(h)(t)\|_{L^2}^2 = 2\Re\left(i\int_0^t\int_{\mathbf{R}}(\Phi h)(s,x)\overline{S(h)(s,x)}dxds\right)$$

and after some computations,

$$R \leq \|S(h)(T)\|_{L^2}^2 \leq 4T\|\Phi\|_{\mathcal{L}(L^2)}^2\int_0^T\|h(s)\|_{L^2}^2 ds$$

- This allows to obtain lower bounds on the L^2 norm of any control allowing to get in the large deviation set
- We deduce upper bounds on the probabilities thanks to the LDP, bounds depend on $\|\Phi\|_{\mathcal{L}(L^2)}^2$

Tails of arrival time, lower bounds, $u_0 = \Psi_A^0$:

The control is searched in the form $h(t,x) = \lambda(t)x$, hence $(\Phi = Id)$:

$$\frac{i\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + |u|^2 u = \lambda(t)xu$$

- then $v_1(t) = \exp(i(\int_0^t \lambda(s)ds)x)u(t)$ is solution of

  $$i\frac{\partial v_1}{\partial t} + \frac{\partial^2 v_1}{\partial x^2} + |v_1|^2 v_1 + \Big(\int_0^t \lambda(s)ds\Big)^2 v_1 + 2i\Big(\int_0^t \lambda(s)ds\Big)\frac{\partial v_1}{\partial x} = 0$$

  and $v_2(t) = \exp(-i\int_0^t(\int_0^s \lambda(\tau)d\tau)^2 ds)v_1(t)$ (Gauge transform) satisfies

  $$i\Big(\frac{\partial v_2}{\partial t} + 2\Big(\int_0^t \lambda(s)ds\Big)\frac{\partial v_2}{\partial x}\Big) + \frac{\partial^2 v_2}{\partial x^2} + |v_2|^2 v_2 = 0$$

- it follows easily that

  $$v_3(t,x) = v_2(t, x + 2\int_0^t \int_0^s \lambda(\tau)d\tau ds)$$

  is a solution of the usual NLS equation with initial data Ψ_A^0.

- It is even more convenient to work with

$$i\frac{\partial v}{\partial t} + \partial_x^2 v + |v|^2 v = \lambda(t)\big(x - 2\int_0^t \int_0^s \lambda(\tau)d\tau ds\big)v$$

$\rightsquigarrow$ use another Gauge transform

- using the preceding (reversed) transformations, we find

$$Y(S(h)(T)) = 2\Big(\int_0^T \int_0^s \lambda(\tau)d\tau ds\Big)M(S(h)(T)) = 8A\int_0^T \int_0^t \lambda(s)dsdt$$

(with $Y(\Psi_A^0) = 0$) and

$$\|h\|^2_{L^2(0,T;L^2)} = \frac{\pi^2}{3A}\int_0^T \lambda^2(s)ds$$

- It follows

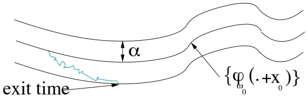
$$\frac{1}{2}\inf_{\substack{h \in L^2(0,T;L^2) \\ Y(S(h)(T)) \geq R}} \|h\|^2_{L^2(0,T;L^2)} \leq \inf_{\substack{\lambda \in L^2(0,T) \\ \int_0^T \int_0^t \lambda(s)dsdt \geq R}} \frac{\pi^2}{6A}\int_0^T \lambda^2(t)dt$$

- the integral constraint gives the factor $1/T^3$
- Again, use an approximation argument for Φ

Remark and open problem

- Same argument (LDP +control problem or energy estimates) has been used to derive bounds on probabilities of exit time of a neighborhood of a modulated soliton for the stochastic KdV equation

$$du + (\partial_x^3 u + u\partial_x u)dt = \varepsilon dW(t,x)$$



- Same problem for stochastic GP equation

$$idu + (\Delta u - |x|^2 u)dt + |u|^{2\sigma} udt = \varepsilon u \circ dB_t$$

but only upper bounds are available : multiplicative noise ⇝ control pb open

G. Ben Arous, M. Gradinaru and M. Ledoux,
"Hölder norms and the support theorem for diffusions,"
Ann. Inst. H. Poincaré, Prob. Stat. **30** (1994) 415-436.

G. Da Prato and J. Zabczyk,
"Stochastic equations in infinite dimensions,"
Encyclopedia of Mathematics and its Applications ; Cambridge University Press : Cambridge, England, 1992.

A. de Bouard and A. Debussche,
"The stochastic nonlinear Schrödinger equation in H^1",
Stochastic Anal. Appl. **21** (2003) 97–126.

A. de Bouard and A. Debussche,
"On the effect of a noise on the solutions of supercritical nonlinear Schrödinger equations,"
Probab. Theory Rel. Fields **123** (2002) 76–96.

A. de Bouard and A. Debussche,
"Blow up for the stochastic nonlinear Schrödinger equation with multiplicative noise,"
Ann. Probab. **33** (2005) 1078–1110.

A. de Bouard and E. Gautier
"Exit problems related to the persistence of solitons for the Korteweg-de Vries equation with small noise,"
Disc. Cont. Dyn. Syst. Serie A. **26** (2010) 857–871.

A. de Bouard and R. Fukuizumi,
"Representation formula for stochastic Schrdinger evolution equations and applications,
to appear in Nonlinearity

A. Debussche,
"Ergodicity results for the stochastic Navier-Stokes equations : an introduction,"
Preprint, 2011.

A. Debussche and L. Di Menza,
"Numerical simulation of focusing stochastic nonlinear Schrödinger equations,"
Physica D **162** (2002) 131–154.

A. Debussche and E. Gautier,
"Small noise asymptotic of the timing jitter in soliton transmission,"
Probab. Theory Relat. Fields **102** (1995) 367–391.

F. Flandoli and D. Gatarek,
"Martingale and stationary solutions for stochastic Navier-Stokes equations,"
Ann. Appl. Probab. **18** (2008) 178–208.

C. Kenig, G. Ponce and L. Vega,
"On the ill-posedness of some canonical dispersive equations,"
Duke Math. J.. **106** (2001) 617–633.

A. Millet and M. Sanz-Sole,
"A simple proof of the support theorem for diffusion processes,"
Séminaire de Proba. XXVIII., LNM 1583, Springer, (1994) 36–48.