Stochastic equations and control theory

A. de Bouard and M. Fuhrman

Ecole Polytechnique and Politecnico di Milano

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Malliavin calculus

Malliavin calculus for scalar Wiener process

$(\Omega, \mathcal{F}, \mathbf{P})$ probability space, $B(t)$, $t \geq 0$ standard brownian motion.
$\mathcal{F}_t = \sigma(B(s), s \in [0, t])$, $\mathcal{F} = \mathcal{F}_T$.

- The Malliavin derivative is a linear, closed, unbounded operator

$$D : L^2(\Omega) \supset dom(D) \to L^2(\Omega \times [0, T]).$$

- For $g \in L^2(\Omega \times [0, T])$, it satisfies

$$D_s \int_0^T g(\sigma)\, dB(\sigma) = \begin{cases} g(s), & s \in [0, T], \\ 0, & s \geq T. \end{cases}$$

- $D_s u = 0$ if $s > t$ and u is $\mathcal{F}_t$-measurable.
  "Proof" : for $u = \int_0^T g(s) dB(s)$ one "must" have $g(s) = 0$ for $s > t$, so $D_s u = g(s) = 0$ for $s > t$.

- We often use the space :

$$\mathbb{D}^{1,2} = \{u \in L^2(\Omega),\ D_s u \in L^2(\Omega \times [0,T])\}$$

  with the norm :

$$\|u\|^2_{\mathbb{D}^{1,2}} = \mathbf{E}(|u|^2) + \int_0^T \mathbf{E}(|D_s u|^2) ds.$$

  (Many generalisations exist).
- The Malliavin derivative satisfies the chain rule :
  If $u \in \mathbb{D}^{1,2}$, $f \in C^1_b(\mathbf{R})$ then $f(u) \in \mathbb{D}^{1,2}$ and

$$D_s f(u) = f'(u)\, D_s u.$$

The integration by part formula :

If $h \in L^2(\Omega \times [0,T])$ is adapted (i.e. $h(t)$ is $\mathcal{F}_f$-measurable) and $u = \int_0^T g(s) dB(s)$ then

$$\mathbf{E}\left(\int_0^T g(s)\, dB(s) \int_0^T h(s)\, dB(s)\right) = \mathbf{E}\int_0^T g(s)h(s)\, ds$$

$$\rightsquigarrow \mathbf{E}\left(u \int_0^T h(s)\, dB(s)\right) = \mathbf{E}\int_0^T D_s u\, h(s)\, ds,$$

This equality extends to all $u \in \mathbb{D}^{1,2}$.

The integration by part formula :

For $u \in \mathbb{D}^{1,2}$, $h \in L^2(\Omega \times [0, T])$ adapted,

$$\mathbf{E}\left(u \int_0^T h(s)\, dB(s)\right) = \mathbf{E} \int_0^T D_s u\, h(s)\, ds,$$

$$\left(u, \int_0^T h(s)\, dB(s)\right)_{L^2(\Omega)} = (Du, h)_{L^2(\Omega\times[0,T])} \qquad (1)$$

$h \in L^2(\Omega\times[0, T])$ adapted $\Rightarrow h \in dom(D^*),\ D^* h = \int_0^T h(s)\, dB(s).$

If $h \in dom(D^*) \subset L^2(\Omega \times [0, T])$ (not adapted) we still denote

$$D^* h = \int_0^T h(s)\, dB(s) \in L^2(\Omega)$$

and call it the Skorohod integral. Then (1) is valid for $u \in \mathbb{D}^{1,2}$, $h \in dom(D^*)$.

We have $h \in dom(D^*)$ if, for instance,

$$\mathbf{E} \int_0^T \int_0^T |D_r h(s)|^2 drds < \infty.$$

Malliavin derivative of the solution of a SDE

Let B be a brownian motion and X be the solution of :

$$dX = f(X)dt + \sigma(X)dB,\ X(0) = x.$$

Equivalently :

$$X(t) = x + \int_0^t f(X(r))dr + \int_0^t \sigma(X(r))dB(r).$$

By the chain rule :

$$\begin{aligned} D_sX(t) &= \int_0^t f'(X(r))D_sX(r)dr + \int_0^t \sigma'(X(r))D_sX(r)dB(r) + \sigma(X(s)) \\ &= \int_s^t f'(X(r))D_sX(r)dr + \int_s^t \sigma'(X(r))D_sX(r)dB(r) + \sigma(X(s)). \end{aligned}$$

$\rightsquigarrow$ $D_sX(t) = \eta_s(t)$ where η_s is the solution of :

$$d\eta_s = f'(X)\eta_s dt + \sigma'(X)\eta_s dB,\ \ \eta_s(s) = \sigma(X(s)).$$

Reminders on cylindrical Wiener process and the Ito integral

- $(\Omega, \mathcal{F}, \mathbf{P})$ probability space, H, K Hilbert spaces with c.o.s. (e_k) and (f_l), B_k a sequence of independent standard brownian motions.
- $W(t)$, $t \geq 0$ cylindrical Wiener process in H, i.e.

$$W(t) = \sum_{k \in \mathbf{N}} e_k B_k(t)$$

- $\mathcal{F}_t = \sigma(B_k(s),\, s \in [0, t],\, k \in \mathbf{N})$, $\mathcal{F} = \mathcal{F}_T$.
- If $\Psi(\omega, t)$ takes values in $HS(H, K)$, is adapted and $\Psi \in L^2(\Omega \times [0, T]; HS(H, K))$, i.e.

$$\mathbf{E} \int_0^T \|\Psi(s)\|^2_{HS(H,K)} ds < \infty$$

then the stochastic integral $I := \int_0^T \Psi(s)\, dW(s)$ is defined and belongs to $L^2(\Omega; K)$, i.e. $\mathbf{E}\|I\|_K^2 < \infty$.

The Malliavin derivative

The Wiener process contains all the independent brownian motions $B_k \rightsquigarrow$ one can differentiate with respect to any of them.
If $\Psi \in L^2(\Omega \times [0,T]; HS(H,K))$:

$$\int_0^t \Psi(s)dW(s) = \sum_{j,k} \int_0^t (\Psi(s)e_k, f_j)_K dB_k(s) f_j$$

and

$$D_s^k \int_0^t \Psi(s)dW(s) = \sum_j (\Psi(s)e_k, f_j)_K f_j = \Psi(s)e_k.$$

This defines an operator $D_s \int_0^t \Psi(s)dW(s)$ on H, for $h = \sum_k h_k e_k \in H$:

$$D_s \int_0^t \Psi(s)dW(s) \cdot h = \sum_k h_k \Psi(s)e_k = \Psi(s) \cdot h.$$

The Malliavin derivative

If $\Psi \in L^2(\Omega \times [0,T]; HS(H,K))$:

$$\int_0^t \Psi(s)dW(s) = \sum_{j,k} \int_0^t (\Psi(s)e_k, f_j)_K dB_k(s) f_j$$

and

$$D_s^k \int_0^t \Psi(s)dW(s) = \sum_j (\Psi(s)e_k, f_j)_K f_j.$$

This defines an operator $D_s \int_0^t \Psi(s)dW(s)$ on H :

$$D_s \int_0^t \Psi(s)dW(s) \cdot h = \psi(s) \cdot h \rightsquigarrow D_s \int_0^t \Psi(s)dW(s) = \psi(s).$$

All the good properties of the Malliavin derivative in dimension one can be transated in this framework.
The Malliavin derivative is computed for instance thanks to the chain rule.

Malliavin derivative of a solution of a SPDE

Let $A = \Delta$ with boundary conditions on a domain $\mathcal{O}$, $H = L^2(\mathcal{O})$, $f : H \to H$ a smooth function (can be relaxed), W a cylindrical Wiener process on H, $\Phi \in \mathcal{L}(H)$ and u the solution of

$$du = (Au + f(u))dt + \Phi dW, \; u(0) = u_0 \in H.$$

- $u(t) = e^{At}u_0 + \int_0^t e^{A(t-r)} f(u(r))dr + \int_0^t e^{A(t-r)}\Phi dW(r)$
- $D_s u(t) = \int_s^t e^{A(t-r)} f'(u(r)) \cdot D_s u(r)dr + e^{A(t-s)}\Phi$
- $D_s u(t) \cdot h = \eta_{s,h}(t)$ is the solution of

$$d\eta_{s,h} = A\eta_{s,h} + f'(u) \cdot \eta_{s,h}, \; \eta_{s,h}(s) = \Phi h.$$

Malliavin derivative of a solution of a SPDE

$$du = (Au + f(u))dt + \Phi dW,\ u(0) = u_0 \in H.$$

- $D_s u(t) \cdot h = \eta_{s,h}(t)$ solution of

$$d\eta_{s,h} = A\eta_{s,h} + f'(u) \cdot \eta_{s,h},\ \eta_{s,h}(s) = \Phi h.$$

- Denote by $U(t,s)$ the evolution operator of the linearized equation : $y_{s,k}(t) = U(t,s)k$ is the solution of

$$dy_{s,k} = Ay_{s,k} + f'(u) \cdot y_{s,k},\ y_{s,k}(s) = k.$$

then

$$D_s u(t) = U(t,s)\Phi$$

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Ergodicity for SPDEs

Transition semigroup

We follow G. Da Prato, J. Zabczyk. Ergodicity for infinite-dimensional systems. Cambridge Univ. Press, 1996.
Let $A = \Delta$ with boundary conditions on a domain $\mathcal{O}$, $H = L^2(\mathcal{O})$, $f : H \to H$ a smooth function (can be relaxed), W a cylindrical Wiener process on H, $\Phi \in \mathcal{L}(H)$ and $u(t, u_0)$ the solution of

$$du(t) = Au(t)\,dt + f(u(t))\,dt + \Phi\,dW(t), \quad u(0) = u_0 \in H$$

For $\phi \in B_b(H)$ define

$$P_t\phi(u_0) = \mathbf{E}\,\phi(u(t, u_0)), \quad u_0 \in H.$$

- P_t is a semigroup on $B_b(H)$: $P_{t+s} = P_t P_s$. (Markov property).
- It is stochastically continuous : if $\phi \in C_b(H)$ then $P_t\phi(u_0) \to \phi(u_0)$, $u_0 \in H$, as $t \to 0$.
- Feller property : if $\phi \in C_b(H)$ then $P_t\phi \in C_b(H)$.

Invariant measure

- For $u_0 \in H$, $t \geq 0$, we denote by $\nu_{u(t,u_0)}$ the law of $u(t,u_0)$ as a random variable on H : for $\Gamma \subset H$ a borel set

$$\nu_{u(t,u_0)}(\Gamma) = \mathbf{P}(u(t,u_0) \in \Gamma) = \mathbf{E}\, 1_\Gamma(u(t,u_0)) = P_t 1_\Gamma(u_0).$$

- For $\phi \in B_b(H)$:

$$P_t\phi(u_0) = \mathbf{E}(\phi(u(t,u_0))) = \int_H \phi(v)\nu_{u(t,u_0)}(dv) = \int_H P_t\phi(v)\delta_{u_0}(dv)$$

$\rightsquigarrow \nu_{u(t,u_0)} = P_t^*\delta_{u_0}$

- More generally, if $u(0)$ is random with law μ, i.e.

$$\mathbf{P}(u(0) \in \Gamma) = \mu(\Gamma),$$

then $u(t,u_0)$ has law $P_t^*\mu$.

Invariant measure

- If $u(0)$ has law μ, then $u(t, u_0)$ has law $P_t^*\mu$.
- μ is called an invariant measure if $P_t^*\mu = \mu$, $t \geq 0$. It is the stochastic analogue of equilibrium point.
- Existence of invariant measure can be proved by classical arguments (Krylov-Bogoliubov theorem) under boundedness or dissipativity assumptions on A, f.

Uniqueness and ergodicity of invariant measure

Let μ be an invariant measure. If it is unique then it is ergodic :

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T P_t\phi(u_0)\, dt = \int_H \phi(v)\, \mu(dv), \quad \phi \in L^2(H, \mu).$$

The Doob-Hasminsky theorem : suppose that for all large $t > 0$

1. P_t is irreducible, i.e. $\mathbf{P}(u(t, u_0) \in \Gamma) = P_t 1_\Gamma(u_0) > 0$ for all $u_0 \in H$ and open $\Gamma \subset H$.
2. P_t is strong Feller, i.e. $\phi \in B_b(H) \Rightarrow P_t\phi \in C_b(H)$.

Then, if an invariant measure μ exists,

1. μ is the unique invariant probability measure.
2. The law of $u(t, u_0)$ converges to μ :

$$\lim_{t\to\infty} \nu_{u(t,u_0)} = \mu.$$

3. μ and all probabilities $\nu_{u(t,u_0)}$ are equivalent.

The Bismut-Elworthy-Li formula

Question : How to prove the Strong Feller property ?

Remark : In finite dimension, if we consider an SDE $dX = f(X)dt + \sigma(X)dB$ then $v(t,x) = P_t\phi(x) = \mathbf{E}(\phi(X(t,x)))$ satisfies :

$$\frac{dv}{dt} = \sum_{i,j} a_{i,j}\partial_{ij}v + \sum_i f_i\partial_i v,\ \ v(0) = \phi.$$

with $a = \sigma\sigma^*$. If a is elliptic, $P_t\phi$ is indeed smooth.

The Bismut-Elworthy-Li formula

Question : How to prove the Strong Feller property ?
We write formally for $h \in H$:

$$DP_t\phi(u_0)\cdot h = \mathbf{E}(D\phi(u(t,u_0))\cdot Du(t,u_0)\cdot h) = \mathbf{E}(D\phi(u(t,u_0))\cdot U(t,0)h)$$

Recall that $U(t,0) = U(t,s)U(s,0)$ and $D_s u(t,u_0) = U(t,s)\Phi$:

$$\begin{aligned}
DP_t\phi(u_0)\cdot h &= \mathbf{E}(D\phi(u(t,u_0))\cdot U(t,s)U(s,0)h) \\
&\quad \text{if } \Phi \text{is invertible} \\
&= \mathbf{E}(D\phi(u(t,u_0))\cdot U(t,s)\Phi\Phi^{-1}U(s,0)h) \\
&= \mathbf{E}(D\phi(u(t,u_0))\cdot D_s u(t,u_0)\Phi^{-1}U(s,0)h) \\
&\quad \text{chain rule} \\
&= \mathbf{E}(D_s\phi(u(t,u_0))\cdot \Phi^{-1}U(s,0)h)
\end{aligned}$$

The Bismut-Elworthy-Li formula

Question : How to prove the Strong Feller property ?

$$DP_t\phi(u_0)\cdot h = \mathbf{E}(D_s\phi(u(t,u_0))\cdot \Phi^{-1}U(s,0)h)$$

Integrate with respect to $s\in[0,t]$:

$$\begin{aligned}
t\, DP_t\phi(u_0)\cdot h &= \int_0^t \mathbf{E}\Big(D_s\phi(u(t,u_0))\cdot \Phi^{-1}U(s,0)h\Big)ds \\
&= \sum_k \int_0^t \mathbf{E}\Big(D_s^k\phi(u(t,u_0))(\Phi^{-1}U(s,0)h, e_k)_H\Big)ds \\
&= \sum_k \mathbf{E}\Big(\phi(u(t,u_0))\int_0^t (\Phi^{-1}U(s,0)h, e_k)_H dB_k(s)\Big) \\
&= \mathbf{E}\left(\phi(u(t,u_0))\int_0^t (\Phi^{-1}U(s,0)h, dW(s)\right)
\end{aligned}$$

Application to the Strong Feller property

$$
\begin{aligned}
t\,DP_t\phi(u_0)\cdot h &= \mathbf{E}\left(\phi(u(t,u_0))\int_0^t(\Phi^{-1}U(s,0)h, dW(s)\right)\\
&\leq \|\phi\|_{B_b(H)}\mathbf{E}\left(\left|\int_0^t(\Phi^{-1}U(s,0)h, dW(s)\right|\right)\\
&\leq \|\phi\|_{B_b(H)}\mathbf{E}\left(\int_0^t\|\Phi^{-1}U(s,0)h\|^2 ds\right)^{1/2}
\end{aligned}
$$

$\rightsquigarrow$ If we are able to control the left hand side, P_t maps $B_b(H)$ to Lipschitz functions.

Remark : Note that the noise is rough for Φ invertible ... In dimension one, we can take $\Phi = Id$. In dimension two or three, it is possible to apply this formula but Φ has to satisfy strong assumptions.

Simple example

Let $A = \Delta$ with boundary conditions on $(0,1)$, $H = L^2(0,1)$, $f : H \to H$ a C_b^1 function, W a cylindrical Wiener process on H and u the solution of

$$du = (Au + f(u))dt + dW, \; u(0) = u_0 \in H.$$

$U(t,0) \cdot h = \eta_h(t)$ is the solution of

$$\frac{d}{dt}\eta_h = A\eta_h + f'(u) \cdot \eta_h, \; \eta_h(0) = h.$$

$\rightsquigarrow$ $\frac{d}{dt}\|\eta\|^2_{L^2(0,1)} \leq L_f \|\eta\|^2_{L^2(0,1)}$ and $\|\eta(t)\|^2_{L^2(0,1)} \leq e^{L_f t}\|h\|^2_{L^2(0,1)}$
We obtain :

$$\begin{aligned} t\, DP_t\phi(u_0) \cdot h &\leq \|\phi\|_{B_b(H)} \mathbf{E} \left(\int_0^t \|\Phi^{-1} U(s,0) h\|^2 ds \right)^{1/2} \\ &\leq \|\phi\|_{B_b(H)} \left(\int_0^t e^{L_f s} \|h\|^2_{L^2(0,1)} ds \right)^{1/2} \\ &\leq C\sqrt{t} \|\phi\|_{B_b(H)} e^{\frac{1}{2} L_f t} \end{aligned}$$

The stochastic Navier-Stokes equation : existence of solutions

Consider the Navier-Stokes equations with stochastic forcing :

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + (u.\nabla)u + \nabla p = f + \varepsilon \xi, \\ \text{div } u = 0, \\ u(t,x) = 0, \ x \in \partial \mathcal{O} \end{cases}$$

with $u(t,x) \in \mathbf{R}^d$, $x \in \mathcal{O}$ bounded domain of $\mathbf{R}^d$, $d = 1, 2, 3$.
Assume that $\xi(t,x)$ is given by $\xi(t,x) = \Phi \partial_t \tilde{W}$ where $\tilde{W}$ is a cylindrical Winer process on $L^2(\mathcal{O})^d$ and Φ is a bounded linear operator on $L^2(\mathcal{O})^d$ that possibly introduces some space correlation.
Let

$$H = \left\{ u \in L^2(\mathcal{O})^d, \ \text{div } u = 0, \ u.n = 0 \text{ on } \partial\Omega \right\}$$

then

$$\mathbf{E}(\xi_k(t,x)\xi_l(s,y)) = c_{k,l}(x,y)\delta_{t-s}, \ k, l = 1, ..., d$$

Assume for simplicity $f = 0$ and $\varepsilon = 1$. Let $\mathcal{P}$ be the Leray projector from $L^2(\mathcal{O})^d$ into H, $A = \mathcal{P}\Delta$, with $D(A) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$ and $b(u, v) = -\mathcal{P}(u.\nabla)v$ be the usual bilinear form acting on divergence free fields u, v.

Consider the NS equation in abstract form

$$\begin{cases} du = \nu A u dt + b(u, u)dt + C^{1/2}dW \\ u(0) = u_0 \end{cases}$$

where now W is a cylindrical Wiener process on H and $C^{1/2}$ is a positive bounded symmetric operator on H.

Remark : We consider additive noise here, i.e. C does not depend on u but generalizations are possible.

The linear equation :

Consider first the linear equation

$$\begin{cases} dZ = \nu AZdt + C^{1/2}dW, \ t \geq 0 \\ Z(0) = 0 \end{cases}$$

The solution is formally written as (stochastic convolution) :

$$Z(t) = \int_0^t e^{\nu A(t-s)} C^{1/2} dW(s)$$

Writing $W = \sum_k e_k B_k \rightsquigarrow$

$$Z(t) = \sum_k \int_0^t e^{\nu A(t-s)} C^{1/2} e_k \, dB_k.$$

This defines a Gaussian random process in K as soon as the series is convergent in K.

More precisely from the Itô isometry,

$$\mathbf{E}(\|Z(t)\|_K^2) = \int_0^t \|e^{\nu A(t-s)} C^{1/2}\|_{HS(H,K)}^2 ds$$

Take $K = D((-A)^\beta) \approx H^{2\beta}(\mathcal{O})$ and assume that $C^{1/2}(-A)^\alpha$ is a bounded operator on H, for some $\alpha, \beta \in \mathbf{R}$, we obtain, with $(e_k)_{k\in\mathbf{N}}$ eigenfunctions of A, $(\lambda_k)_{k\in\mathbf{N}}$ corresponding eigenvalues,

$$\begin{aligned}
&\|e^{\nu A(t-s)} C^{1/2}\|_{HS(H,H^{2\beta})}^2 \\
&\leq \|C^{1/2}(-A)^\alpha\|_{\mathcal{L}(H)}^2 \|(-A)^{-\alpha+\beta} e^{\nu A(t-s)}\|_{HS(H)}^2 \\
&= \|C^{1/2}(-A)^\alpha\|_{\mathcal{L}(H)}^2 \sum_k \lambda_k^{-2\alpha+2\beta} e^{-2\nu\lambda_k(t-s)}
\end{aligned}$$

For $\beta = 0$, the series converges iff $\sum_k \lambda_k^{-2\alpha-1}$ converges, i.e. $\alpha > \frac{d-2}{4}$ (since $\lambda_k \sim ck^{2/d}$)
In this case, $Z \in C([0,T]; H)$.

- For $d = 1$, one can take $C = Id$ (space time white noise)
- For $d = 2$, need $C^{1/2}(-A)^\alpha$ bounded for some $\alpha > 0$.

The nonlinear equation, case $d = 2$:

In the additive case, we easily replace the stochastic equation

$$(SNS)\begin{cases} du = \nu A u dt + b(u,u)dt + C^{1/2}dW \\ u(0) = u_0 \end{cases}$$

by a random equation by setting $u(t,x) = v(t,x) + Z(t)$ where Z is the stochastic convolution (note that $Z \in C([0,T];H)$) :

$$\begin{cases} \frac{\partial v}{\partial t} = Av + b(v+Z, v+Z) \\ v(0) = u_0 \end{cases}$$

Theorem : Assume $C^{1/2}(-A)^{\alpha} \in \mathcal{L}(H)$ with $\alpha > 1/4$, and $u_0 \in H$; then there is a unique solution u of (SNS) such that $v = u - Z$ has paths a.s. in $C([0,T];H) \cap L^2(0,T;D(A^{1/2}))$; u has paths a.s. in $C([0,T];H)$

If $\alpha > 1$ and $u_0 \in D(A^{1/2})$ than u and v have paths a.s. in $C([0,T];D(A^{1/2}))$

Remark : In the preceding situation, $v \in C^1([0,T]; D(A^{-1}))$; however, u is not more regular in time than Z, i.e. $u \in C^\alpha([0,T]; D(A^{-1}))$, only for $\alpha < 1/2$.

Proof of thm : Standard Galerkin approximation : P_m projector on $\mathrm{span}(e_1, ..., e_m)$

$$\begin{cases} du_m = \nu A u_m + P_m b(u_m, u_m)dt + P_m C^{1/2} dW \\ u_m(0) = P_m u_0 \end{cases}$$

then $u_m = Z_m + v_m$ with

$$\begin{cases} dZ_m = \nu A Z_m dt + P_m C^{1/2} dW \\ Z_m(0) = 0 \end{cases}$$

i.e.

$$Z_m(t) = \int_0^t e^{\nu A(t-s)} P_m C^{1/2} dW(s)$$

Then, **P** a.s., there is a unique $v_m \in C([0,T];H)$ satisfying

$$\frac{dv_m}{dt} = \nu A v_m + b(v_m + Z_m, v_m + Z_m)$$

and taking the inner product with v_m in H :

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}|v_m(t)|_H^2 + \nu|(-A)^{1/2}v_m(t)|_H^2\\ &= b(v_m + Z_m, Z_m), v_m) = -(b(v_m + Z_m, v_m), Z_m)\\ &\leq C|v_m + Z_m|_{L^4}|v_m|_{H^1}|Z_m|_{L^4}\end{aligned}$$

by Gagliardo Nirenberg inequalities $|v|_{L^4} \leq C|v|_H^{1/2}|(-A)^{1/2}v|_H^{1/2}$ and embedding $D((-A)^{1/2}) \subset L^4$, we deduce :

$$\frac{d}{dt}|v_m|_H^2 + \nu|(-A)^{1/2}v_m(t)|_H^2 \leq C|(-A)^{1/4}Z_m|_H^4(1 + |v_m|_H^2)$$

and by Gronwall Lemma, v_m is bounded in $C([0,T];H) \cap L^2(0,T;D(A^{1/2}))$

Remark : $C^{1/2}(-A)^{\alpha}$ bounded with $\alpha > 1/4$
$\rightsquigarrow Z \in C([0,T];D(A^{1/4}))$

Moreover, $\frac{dv_m}{dt}$ is a.s. bounded in $C([0,T];D(A^{-1}))$

$\rightsquigarrow$ for a.e. $\omega \in \Omega$, the sequence v_m is relatively compact in $C([0,T];D(A^s)) \cap L^2(0,T;D(A^{1/2+s}))$ for any $s < 0$

One can then pass to the limit in the equation and obtain the solution in the classical way.

Uniqueness (same estimate) ensures the measurability with respect to ω : actually, v is a Borelian function of $Z \rightsquigarrow v$ is an adapted process (depends only on the past).

Remark on the 1-D case (Burgers) :

Same setting :

$$\begin{cases} \frac{dv}{dt} = Av + b(v+Z, v+Z) \\ v(0) = u_0 \end{cases}$$

- A : Laplace operator on $(0,1)$ with Dirichlet boundary conditions
- $Z(t) = \int_0^t e^{-A(t-s)} dW(s)$ with W : cylindrical Wiener process on $H = L^2(0,1)$ ($\rightsquigarrow$ space-time white noise)
- $b(u,v) = u(\partial_x v)$ for $u, v \in H$.

then there is a unique solution v, a.s. in $C([0,T];H)$ $\cap L^2(0,T;D(A^{1/2}))$

Proof : Same estimates as above, but uses the fact that Z is a.s. in $C^\alpha([0,T]\times(0,1))$ for any $\alpha < 1/4 \rightsquigarrow Z \in L^\infty(0,T;L^4)$ a.s

Remarks on the 3-D case :

- No uniqueness $\rightsquigarrow$ preceding arguments do not work
- If $C^{1/2}(A)^\alpha$ is a bounded operator in H with $\alpha > 5/4$ and if $u_0 \in D(A)$ then there exist a random time $T(u_0)$ and unique local solution a.s. in $C([0, T(u_0)); D(A))$
- If $Tr(C) = \sum_k (Ce_k, e_k) < +\infty$, then there is a weak global solution (solution in law) i.e. there exist $(\Omega, \mathcal{F}, \mathbf{P}, (W(t))_{t\geq 0})$ and a solution u of (SNS) with paths a.s. in $L^2(0, T; D(A^{1/2}) \cap C([0, T]; D(A^s))$ for any $s < 0$, for any $T > 0$
  Based on compactness method (tightness) for the family of laws of approximate solutions (Flandoli, Gatarek, 1995)
  Adaptivity : open problem
  Assumption $Tr(C) < +\infty$ may be relaxed
- Refined results (e.g. there exists such a solution which is a Markov process) and large literature on ergodicity results (G. Da Prato & AD, F. Flandoli & M. Romito...)