Stochastic equations and control theory

A. de Bouard and M. Fuhrman

Ecole Polytechnique and Politecnico di Milano

Control of PDEs, interactions and application challenges,
CIRM,5-9 November 2012

Reminders

Malliavin derivative

- $(\Omega, \mathcal{F}, \mathbf{P})$ probability space, H, K Hilbert spaces with c.o.s. (e_k) and (f_l), B_k independent standard brownian motions.
- W cylindrical Wiener process in H : $W(t) = \sum_{k \in \mathbf{N}} e_k B_k(t)$
- If $\Psi(\omega, t)$ takes values in $HS(K, H)$, is adapted and $\mathbf{E} \int_0^T \|\Psi(s)\|^2_{HS(K,H)} ds < \infty$, then :

$$\int_0^t \Psi(s) dW(s) = \sum_{j,k} \int_0^t (\Psi(s) e_k, f_j)_K dB_k(s) f_j$$

and

$$D_s^k \int_0^t \Psi(s) dW(s) = \sum_j (\Psi(s) e_k, f_j)_K f_j = \Psi(s) e_k.$$

and

$$D_s \int_0^t \Psi(s) dW(s) = \sum_k h_k \Psi(s) e_k = \Psi(s).$$

The Malliavin derivative is computed e.g. by the chain rule.

Malliavin derivative of a solution of a SPDE

Let $A = \Delta$ with boundary conditions on a domain $\mathcal{O}$, $H = L^2(\mathcal{O})$, $f : H \to H$ a smooth function (can be relaxed), W a cylindrical Wiener process on H, $\Phi \in \mathcal{L}(H)$ and u the solution of

$$du = (Au + f(u))dt + \Phi dW, \; u(0) = u_0 \in H.$$

- $u(t) = e^{At}u_0 + \int_0^t e^{A(t-r)} f(u(r))dr + \int_0^t e^{A(t-r)}\Phi dW(r)$
- $D_s u(t) = \int_s^t e^{A(t-r)} f'(u(r)) \cdot D_s u(r)dr + e^{A(t-s)}\Phi$
- $D_s u(t) \cdot h = U(t,s)\Phi h = \eta_{s,h}(t)$ is the solution of

$$d\eta_{s,h} = A\eta_{s,h} + f'(u) \cdot \eta_{s,h}, \; \eta_{s,h}(s) = \Phi h.$$

Transition semigroup

Let $A = \Delta$ with boundary conditions on a domain $\mathcal{O}$, $H = L^2(\mathcal{O})$, $f : H \to H$ a smooth function (can be relaxed), W a cylindrical Wiener process on H, $\Phi \in \mathcal{L}(H)$ and $u(t, u_0)$ the solution of

$$du(t) = Au(t)\, dt + f(u(t))\, dt + \Phi\, dW(t), \quad u(0) = u_0 \in H$$

- For $\phi \in B_b(H)$ define the transition semigroup :

$$P_t\phi(u_0) = \mathbf{E}\, \phi(u(t, u_0)), \quad u_0 \in H.$$

- If $u(0)$ is random with law μ, then $u(t, u_0)$ has law $P_t^* \mu$.
- μ is called an invariant measure if $P_t^* \mu = \mu$, $t \geq 0.$ It is the stochastic analogue of equilibrium point.
- Existence of invariant measure can be proved by classical arguments (Krylov-Bogoliubov theorem) under boundedness or dissipativity assumptions on A, f.

Uniqueness and ergodicity of invariant measure

Let μ be an invariant measure. If it is unique then it is ergodic :

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T P_t\phi(u_0)\, dt = \int_H \phi(v)\, \mu(dv), \quad \phi \in L^2(H, \mu).$$

The Doob-Hasminsky theorem : suppose that for all large $t > 0$

1. P_t is irreducible, i.e. $\mathbf{P}(u(t, u_0) \in \Gamma) = P_t 1_\Gamma(u_0) > 0$ for all $u_0 \in H$ and open $\Gamma \subset H$.
2. P_t is strong Feller, i.e. $\phi \in B_b(H) \Rightarrow P_t\phi \in C_b(H)$.

Then, if an invariant measure μ exists,

1. μ is the unique invariant probability measure.
2. The law of $u(t, u_0)$ converges to μ :

$$\lim_{t\to\infty} \nu_{t,u_0} = \mu.$$

3. μ and all probabilities $\nu_{u(t,u_0)}$ are equivalent.

The Bismut-Elworthy-Li formula

$$
\begin{aligned}
DP_t\phi(u_0)\cdot h &= \mathbf{E}(D\phi(u(t,u_0))\cdot U(t,s)U(s,0)h) \\
&\quad \text{if } \Phi \text{is invertible} \\
&= \mathbf{E}(D\phi(u(t,u_0))\cdot U(t,s)\Phi\Phi^{-1}U(s,0)h) \\
&= \mathbf{E}(D\phi(u(t,u_0))\cdot D_s u(t,u_0)\Phi^{-1}U(s,0)h) \\
&\quad \text{chain rule} \\
&= \mathbf{E}(D_s\phi(u(t,u_0))\cdot \Phi^{-1}U(s,0)h)
\end{aligned}
$$

Integrate with respect to $s \in [0,t]$:

$$
\begin{aligned}
t\, DP_t\phi(u_0)\cdot h &= \int_0^t \mathbf{E}\left(D_s\phi(u(t,u_0))\cdot\Phi^{-1}U(s,0)h\right)ds \\
&= \mathbf{E}\left(\phi(u(t,u_0))\int_0^t(\Phi^{-1}U(s,0)h, dW(s)\right)
\end{aligned}
$$

Application to the Strong Feller property

$$
\begin{aligned}
t\,DP_t\phi(u_0)\cdot h &= \mathbf{E}\left(\phi(u(t,u_0))\int_0^t (\Phi^{-1}U(s,0)h, dW(s)\right)\\
&\le \|\phi\|_{B_b(H)}\mathbf{E}\left(\left|\int_0^t (\Phi^{-1}U(s,0)h, dW(s)\right|\right)\\
&\le \|\phi\|_{B_b(H)}\mathbf{E}\left(\int_0^t \|\Phi^{-1}U(s,0)h\|^2 ds\right)^{1/2}
\end{aligned}
$$

$\rightsquigarrow$ If we are able to control the left hand side, P_t maps $B_b(H)$ to Lipschitz functions.

The stochastic Navier-Stokes equation, case $d = 2$:

Consider the NS equation in abstract form

$$\begin{cases} du = \nu A u dt + b(u,u)dt + C^{1/2} dW \\ u(0) = u_0 \end{cases}$$

where W is a cylindrical Wiener process on H and $C^{1/2}$ is a positive bounded symmetric operator on H.

Remark : We consider additive noise here, i.e. C does not depend on u but generalizations are possible.

The stochastic Navier-Stokes equation, case $d = 2$:

In the additive case, we easily replace the stochastic equation

$$(SNS)\begin{cases} du = \nu A u dt + b(u,u)dt + C^{1/2}dW \\ u(0) = u_0 \end{cases}$$

by a random equation by setting $u(t,x) = v(t,x) + Z(t)$ where Z is the stochastic convolution (note that $Z \in C([0,T];H)$) :

$$\begin{cases} \frac{\partial v}{\partial t} = Av + b(v+Z, v+Z) \\ v(0) = u_0 \end{cases}$$

Theorem : Assume $C^{1/2}(-A)^{\alpha} \in \mathcal{L}(H)$ with $\alpha > 1/4$, and $u_0 \in H$; then there is a unique solution u of (SNS) such that $v = u - Z$ has paths a.s. in $C([0,T];H) \cap L^2(0,T;D(A^{1/2}))$; u has paths a.s. in $C([0,T];H)$
If $\alpha > 1$ and $u_0 \in D(A^{1/2})$ than u and v have paths a.s. in $C([0,T];D(A^{1/2}))$

The stochastic Navier-Stokes equation : ergodicity for rough noise

Consider the NS equation in abstract form

$$\begin{cases} du = \nu A u dt + b(u,u)dt + C^{1/2}dW \\ u(0) = u_0 \end{cases}$$

where now W is a cylindrical Wiener process on H and $C^{1/2}$ is a positive bounded symmetric operator on H.
We have a solution under the assumption that $C^{1/2}(-A)^{\alpha} \in \mathcal{L}(H)$ with $\alpha > 1/4$. We want to apply the Bismut-Elworthy-Li formula :

$$\begin{aligned} DP_t\phi(u_0)\cdot h &= \frac{1}{t}\mathbf{E}\left(\phi(u(t,u_0))\int_0^t (C^{-1/2}U(s,0)h, dW(s)\right) \\ &\leq \frac{1}{t}\|\phi\|_{B_b(H)}\mathbf{E}\left(\left|\int_0^t (C^{-1/2}U(s,0)h, dW(s)\right|\right) \\ &\leq \frac{1}{t}\|\phi\|_{B_b(H)}\mathbf{E}\left(\int_0^t \|C^{-1/2}U(s,0)h\|^2 ds\right)^{1/2} \end{aligned}$$

Consider the NS equation in abstract form

$$\begin{cases} du = \nu A u dt + b(u,u)dt + C^{1/2}dW \\ u(0) = u_0 \end{cases}$$

We want to apply the Bismut-Elworthy-Li formula :

$$DP_t\phi(u_0)\cdot h \quad \leq \frac{1}{t}\|\phi\|_{B_b(H)}\mathbf{E}\left(\int_0^t \|C^{-1/2}U(s,0)h\|^2 ds\right)^{1/2}$$

Recall that $U(t,0)\cdot h = \eta_h(t)$ is the solution of

$$\frac{d}{dt}\eta_h = A\eta_h + b(u,\eta_h) + b(\eta_h,u),\ \eta_h(0) = h.$$

$\rightsquigarrow \frac{d}{dt}\|\eta\|^2_{L^2(0,1)} + \|\nabla\eta\|^2_{L^2(0,1)} \leq c\|\nabla u\|^2_{L^2(0,1)}\|\eta\|^2_{L^2(0,1)}$

and

$$\|\eta(t)\|^2_{L^2(0,1)} + \int_0^t \|\nabla\eta\|^2_{L^2(0,1)}ds \leq e^{c\int_0^t \|\nabla u\|^2_{L^2(0,1)}ds}\|h\|^2_{L^2(0,1)}$$

Consider the NS equation in abstract form

$$\begin{cases} du = \nu A u dt + b(u,u)dt + C^{1/2}dW \\ u(0) = u_0 \end{cases}$$

We want to apply the Bismut-Elworthy-Li formula :

$$DP_t\phi(u_0)\cdot h \quad \le \frac{1}{t}\|\phi\|_{B_b(H)}\mathbf{E}\left(\int_0^t \|C^{-1/2}U(s,0)h\|^2 ds\right)^{1/2}$$

We deduce :

$$\|U(t,0)h\|^2_{L^2(0,1)} + \int_0^t \|\nabla U(s,0)h\|^2_{L^2(0,1)}ds \le e^{c\int_0^t \|\nabla u\|^2_{L^2(0,1)}ds}\|h\|^2_{L^2(0,1)}$$

and obtain :

$$DP_t\phi(u_0)\cdot h \quad \le \frac{1}{t}\|\phi\|_{B_b(H)}\mathbf{E}\left(e^{c\int_0^t \|\nabla u\|^2_{L^2(0,1)}ds}\right)^{1/2}\|h\|_{L^2(0,1)}$$

if $\|C^{-1/2}k\| \le \|\nabla k\| \sim \|(-A)^{1/2}k\|$.

Consider the NS equation in abstract form

$$\begin{cases} du = \nu A u dt + b(u,u)dt + C^{1/2} dW \\ u(0) = u_0 \end{cases}$$

We have by Bismut-Elworthy-Li formula :

$$DP_t\phi(u_0) \cdot h \quad \leq \frac{1}{t}\|\phi\|_{B_b(H)} \mathbf{E}\left(e^{c\int_0^t \|\nabla u\|^2_{L^2(0,1)} ds}\right)^{1/2} \|h\|_{L^2(0,1)}$$

- The two conditions $\|C^{-1/2}k\| \leq \|\nabla k\| \sim \|(-A)^{1/2}k\|$ and $C^{1/2}(-A)^{\alpha} \in \mathcal{L}(H)$ with $\alpha > 1/4$ are compatible ! But the noise has to be very rough in space.
- The quantity inside the expectation has an infinite expectation ... Needs a truncation argument.
  This is the idea of the proof done by F. Flandoli and B. Maslowski.

The stochastic Navier-Stokes equation : ergodicity for smooth noise

- Flandoli and Maslowski's result has the drawback that it requires very rough noise. In practice, the noise is often very smooth and can even be finite dimensional.
- Kuksin, Shirikyan, Bricmont, Kupiainen, Lefevere, E, Mattingly, Sinai, Hairer ... have develop different ideas which do not require any assumption of roughness on the noise.
- We present an approach of these result following an argument developed by Hairer and Mattingly.

Beyond the Bismut-Elworthy-Li formula

Let us go back to the proof of this formula :

$$DP_t\phi(u_0)\cdot h = \mathbf{E}(D\phi(u(t,u_0))\cdot Du(t,u_0)\cdot h) = \mathbf{E}(D\phi(u(t,u_0))\cdot U(t,0)h)$$

We use $U(t,0) = U(t,s)U(s,0)$

$$DP_t\phi(u_0)\cdot h = \mathbf{E}(D\phi(u(t,u_0))\cdot U(t,s)U(s,0)h)$$

But $D_s u(t,u_0) = U(t,s)C^{1/2}$ and since $C^{1/2}$ is not invertible, we do not know if $U(s,0)h$ is in the range of $C^{1/2}$. Assume that this is the case and take $k(s)$ such that :

$$U(s,0)h = C^{1/2}k(s)$$

Then

$$DP_t\phi(u_0)\cdot h = \mathbf{E}(D\phi(u(t,u_0))\cdot U(t,s)C^{1/2}k(s)) = \mathbf{E}(D_s\phi(u(t,u_0))\cdot k(s))$$

$$DP_t\phi(u_0)\cdot h = \mathbf{E}(D\phi(u(t,u_0))\cdot Du(t,u_0)\cdot h) = \mathbf{E}(D\phi(u(t,u_0))\cdot U(t,0)h)$$

Assume that there exists $k(s)$ such that :

$$U(s,0)h = C^{1/2}k(s)$$

Then

$$DP_t\phi(u_0)\cdot h = \mathbf{E}(D\phi(u(t,u_0))\cdot U(t,s)C^{1/2}k(s)) = \mathbf{E}(D_s\phi(u(t,u_0))\cdot k(s))$$

We integrate with respect to $s \in [0,T]$:

$$\begin{aligned} t\,DP_t\phi(u_0)\cdot h &= \int_0^t \mathbf{E}(D\phi(u(t,u_0))\cdot U(t,s)C^{1/2}k(s))ds \\ &= \int_0^t \mathbf{E}(D_s\phi(u(t,u_0))\cdot k(s))ds \\ &= \mathbf{E}\left(\phi(u(t,u_0))\int_0^t (k(s),dW(s))\right) \end{aligned}$$

We see that in fact we need :

$$\int_0^t U(t,s)C^{1/2}k(s))ds = tU(t,0)h$$

Set $v(s) = \frac{1}{t}k(s)$. We need :

$$U(t,0)h - \int_0^t U(t,s)C^{1/2}v(s))ds = 0$$

The left hand side is the value a time t of the solution of :

$$\frac{d}{dt}\eta = A\eta + b(u,\eta) + b(\eta,u) - C^{1/2}v, \quad \eta(0) = h$$

In summary, we are led to a controllability problem. We need to find a control v and then use it in

$$DP_t\phi(u_0)\cdot h = \mathbf{E}\left(\phi(u(t,u_0))\int_0^t (v(s), dW(s))\right)$$

- For general C, this might be a very difficult control problem.
- We need to bound the stochastic integral : the control has to be square integrable.
- It is even worse because, the control may not be adapted : the stochastic integral is a stochastic integral and is bounded by the quantity :

$$\mathbf{E}\int_0^t \|v(s)\|^2 ds + \mathbf{E}\int_0^t\int_0^t \|D_r v(s)|^2 drds.$$

We relax the problem and ask that
$\eta(t) = U(t,0)h - \int_0^t U(t,s)C^{1/2}v(s))ds$ is small. Then :

$$
\begin{aligned}
& DP_t\phi(u_0)\cdot h \\
&= \mathbf{E}(D\phi(u(t,u_0))\cdot U(t,0)h) \\
&= \int_0^t \mathbf{E}(D\phi(u(t,u_0))\cdot U(t,s)C^{1/2}v(s))ds + \mathbf{E}(D\phi(u(t,u_0))\eta(t)) \\
&= \int_0^t \mathbf{E}(D_s\phi(u(t,u_0))\cdot v(s))ds + \mathbf{E}(D\phi(u(t,u_0))\eta(t)) \\
&= \mathbf{E}\left(\phi(u(t,u_0))\int_0^t (v(s),dW(s))\right) + \mathbf{E}(D\phi(u(t,u_0))\eta(t))
\end{aligned}
$$

$$
\begin{aligned}
\rightsquigarrow \qquad & |DP_t\phi(u_0)\cdot h| \\
& \leq \|\phi\|_{B_b(H)}\mathbf{E}\left(\left|\int_0^t (v(s),dW(s))\right|\right) + \epsilon_t\|D\phi\|_{B_b(H)}
\end{aligned}
$$

If $\eta(t)$ is small.

Theorem (Hairer-Mattingly)
Assume that there exists $\delta_{t_n} \to 0$ as $t_n \to \infty$, and a locally bounded function $c(\|u_0\|)$

$$|DP_{t_n}\varphi(u_0)| \leq c(\|u_0\|)\left(\|\varphi\|_{B_b(H)} + \delta_{t_n}\|D\varphi\|_{B_b(H)}\right)$$

then $(P_t)_{t\geq 0}$ is Asymptotic Strong Feller. If moreover there exists $\bar{u} \in H$ which is in the support of every invariant measure then there exists at most one invariant measure.

The case of a large number of excited modes

We simplify the equation, replace the nonlinearity by a Lipschitz function and assume that $C = P_N$ where P_N is the eigenprojector of A corresponding to the first N eigenvalues :

$$du = (Au + F(u))dt + P_N dW$$

The control problem becomes

$$\frac{d}{dt}\eta = A\eta + F'(u)\eta - P_N v, \quad \eta(0) = h.$$

We split $\eta = \eta_1 + \eta_2$ with $\eta_1 = P_N\eta$, $\eta_2 = (I - P_N)\eta$. We choose η_1 as the solution of

$$\frac{d}{dt}\eta_1 = -\frac{\eta_1}{|\eta_1|}, \quad \eta_1(0) = P_N h.$$

Then $\eta_1(t) = 0$ for $t \geq 1$ and is very easy to bound.

The case of a large number of excited modes

$$du = (Au + F(u))dt + P_N dW$$

$$\frac{d}{dt}\eta = A\eta + F'(u)\eta - P_N v, \quad \eta(0) = h.$$

$\eta = \eta_1 + \eta_2$ with $\eta_1 = P_N\eta$, $\eta_2 = (I - P_N)\eta$.

$$\frac{d}{dt}\eta_1 = -\frac{\eta_1}{|\eta_1|}, \quad \eta_1(0) = P_N h.$$

Then η_2 is the solution of

$$\frac{d}{dt}\eta_2(t) = A\eta_2(t) + (I - P_N)F'(u(t))(\eta_1(t) + \eta_2(t)), \; \eta(0) = (I - P_N)h.$$

and the control v is

$$v(t) = -\frac{d}{dt}\eta_1(t) + A\eta_1(t) + P_N F'(u(t))(\eta_1(t) + \eta_2(t)),$$

Note that it is adapted ! It remains to show that $|\eta_2(t)|$ is small.

$$\frac{d}{dt}\eta_2(t) = A\eta_2(t)+(I-P_N)F'(u(t))(\eta_1(t)+\eta_2(t)),\ \eta_2(0) = (I-P_N)h.$$

Take the scalar product with $\eta_2(t)$ and get the energy inequality

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\eta_2(t)\|^2 + \|(-A)^{1/2}\eta_2(t)\|^2 \\
&\quad = \Big((I - P_N)F'(u(t))(\eta_1(t) + \eta_2(t)), \eta_2(t)\Big) \\
&\quad \le C_1 + C_2\,\|\eta_2(t)\|^2.
\end{aligned}$$

We have $\|(-A)^{1/2}\eta_2(t)\|^2 \ge \lambda_{N+1}\|\eta_2(t)\|^2$,
Taking N so large that $\lambda_{N+1} \ge 2C_2$ we obtain

$$\frac{d}{dt}\|\eta_2(t)\|^2 + (2\lambda_{N+1} - C_2)\|\eta_2(t)\|^2 \le 2C_1.$$

By the Gronwall lemma,

$$\|\eta_2(t)\|^2 \le \frac{2C_1}{2\lambda_{N+1} - C_2} + e^{-(2\lambda_{N+1}-C_2)t}\|h\|^2 \le \delta,$$

for δ small if $\|h\| \le 1$ and N is large. It is then easy to bound v and to conclude by iteration of this argument.

The case of a small number of excited modes

- Let
$$A_{0,t}v = \int_0^t U(t,s)C^{1/2}v(s)ds$$
- Ideally, we want v such that $A_{0,t}v = U(t,0)h$.
- Let $A_{0,t}^*$ be its adjoint (which is then defined from H to $L^2(0,t;H)$) and assume that $M_t = A_{0,t}A_{0,t}^*$ is invertible then we could choose
$$v = A_{0,t}^*(A_{0,t}A_{0,t}^*)^{-1}U(t,0)h.$$
- M_t is called the Malliavin matrix and is invertible in finite dimension under the Hörmander condition.
- It can be shown that if few modes are excited, M_t is injective for the stochastic Navier-Stokes equation but it is not known if is indeed invertible.

The case of a small number of excited modes

- Let

$$A_{0,t}v = \int_0^t U(t,s)C^{1/2}v(s)ds$$

- Ideally, we want v such that $A_{0,t}v = U(t,0)h$,
- Assume that $M_t = A_{0,t}A_{0,t}^*$ is invertible then we could choose

$$v = A_{0,t}^*(A_{0,t}A_{0,t}^*)^{-1}U(t,0)h.$$

- If few modes are excited, M_t is injective for the stochastic Navier-Stokes equation but it is not known if is invertible.
- For finite dimensional systems, this is sufficient to prove ergodicity.
- In the linear case, $A_{0,t}v = \int_0^t e^{A(t-s)}C^{1/2}v(s)ds$, $A_{0,t}^*w(s) = C^{1/2}e^{A(t-s)}w$ and $M_t = \int_0^t e^{A(t-s)}Ce^{A(t-s)}ds$. For degenerate C, it cannot be invertible. This is natural since if invertibility is true, this is due to nonlinear effect.

The case of a small number of excited modes

Let

$$A_{0,t}v = \int_0^t U(t,s)C^{1/2}v(s)ds$$

and $A_{0,t}^*$ be its adjoint which is then defined from H to $L^2(0,t;H)$. Ideally, we want v such that $A_{0,t}v = U(t,0)h$. Assume that $M_t = A_{0,t}A_{0,t}^*$ is invertible then we could choose

$$v = A_{0,t}^*(A_{0,t}A_{0,t}^*)^{-1}U(t,0)h.$$

As seen before, approximate controllability is easier. Hairer and Mattingly have shown that the following choice is OK :

$$\begin{cases} v(t) = A_{n,n+1/2}(A_{n,n+1/2}A_{n,n+1/2}^* + \beta)^{-1}U(n+1/2,n)\eta(n), \\ \hfill t \in [n, n+1/2] \\ \\ v(t) = 0, \ t \in [n+1/2, n+1]. \end{cases}$$

Very long computations .... The control is not adapted ...